solutions to

DUMBELLS

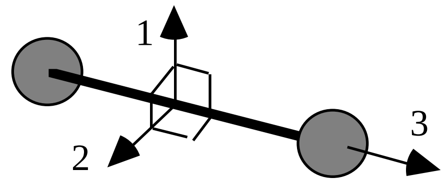
Worksheet 1

by Philip J. Siemens

TOPICS page

A. CALCULATION: SIMPLE DUMBELL

A uniform rod of mass m, length L and diameter d connects the centers of two spheres, each of mass M and radius R.



The spheres have a uniform density ρ , and the diameter of the rod is much less than the radius of the spheres, $d \ll R$.

A coordinate system with 3 orthogonal axes has its origin at the center of the rod. The third axis of the coordinate system coincides with the direction of the rod.

A.1. Longitudinal axis.

Find the moment of inertia about the longitudinal axis (axis 3 in the drawing)

The density of the rod is $\rho_{rod} = \dfrac{m}{\text{volume}} = \dfrac{m}{\pi(d/2)^2 L}$.

The density of the sphere is $\rho = \dfrac{M}{\text{volume}} = \dfrac{M}{\frac{4}{3}\pi R^3}$.

<u>For the rod</u>, choose cylindrical coordinates with z along the length of the rod:

$$I_{long} = \int d^3r \; r^{\,2} \rho(\boldsymbol{r}) = \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\varphi \int_0^{d/2} r\, dr \; r^2 \rho_{rod} = L\, 2\pi \frac{1}{4} \left(\frac{d}{2}\right)^4 \rho_{rod} = \frac{md^2}{8}$$

<u>For each sphere</u>, choose spherical coordinates with $\theta = 0$ along the z axis:

$$I_{long} = \int d^3r \; r^{\,2} \rho(\boldsymbol{r}) = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^{R} r^2 \, dr \; (r \sin\theta)^2 \rho$$

$$= 2\pi \int_{-1}^{1} d(\cos\theta)\,(1-\cos^2\theta)\, \frac{1}{5} R^5 \rho = 2\pi \left(2 - \frac{2}{3}\right) \frac{1}{5} R^5 \rho = \frac{2}{5} MR^2$$

<u>Altogether</u>, $I_{long} = I_{long}(\text{rod}) + 2I_{long}(\text{sphere}) = \frac{1}{8} m\, d^2 + \frac{4}{5} M\, R^2$

A.2. Transverse axes

Find the moment of inertia about each of the transverse axes (axes 1 and 2) .

The moments will be the same, by symmetry:
nothing happens if we relabel the axes $1 \leftrightarrow 2$.

For the rod, use the same coordinates, but now $r = \sqrt{y^2+z^2} = \sqrt{(r \sin \varphi)^2+z^2}$:

$$I_{\text{trans}} = \int d^3r \; r^2 \; \rho(\mathbf{r}) = \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\varphi \int_0^{d/2} r \, dr \; \left((r \sin \varphi)^2+z^2\right) \rho_{\text{rod}} \, .$$

Integrating term by term, we find

$$I_{\text{trans}} = \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\varphi \sin^2\varphi \int_0^{d/2} dr \; r^3 \; \rho_{\text{rod}} + \int_{-L/2}^{L/2} dz \, z^2 \int_0^{2\pi} d\varphi \int_0^{d/2} r \, dr \; \rho_{\text{rod}}$$

$$= L \, \pi \frac{1}{4} \left(\frac{d}{2}\right)^4 \rho_{\text{rod}} + \frac{2}{3} \left(\frac{L}{2}\right)^3 2\pi \frac{1}{2} \left(\frac{d}{2}\right)^2 \rho_{\text{rod}}$$

$$= m \left(\frac{1}{16} d^2 + \frac{1}{12} L^2 \right) .$$

For the sphere, the moment is the same about every axis **through the center**, by symmetry, because every direction is the same:

$$I'_{\text{trans}} = I'_{\text{long}} = I_{\text{long}} = \frac{2}{5} M R^2 \, .$$

But the center of each sphere is a distance $\frac{1}{2} L$ from the center of the rod, so

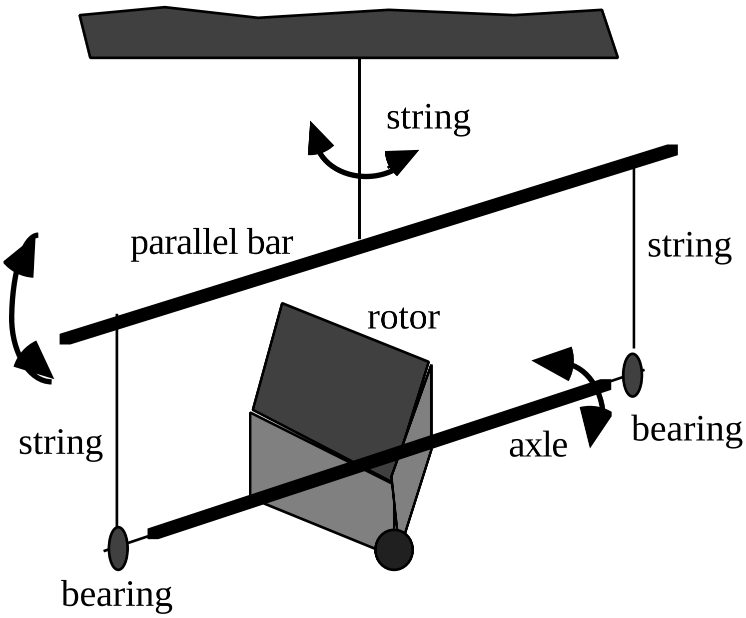
$$I_{\text{trans}} = I'_{\text{trans}} + M \left(\frac{L}{2}\right)^2 = M \left(\frac{1}{4} L^2 + \frac{2}{5} R^2\right) .$$

Altogether,

$$I_{\text{trans}} = I_{\text{trans}}(\text{rod}) + 2 I_{\text{trans}}(\text{sphere}) = m \left(\frac{1}{16} d^2 + \frac{1}{3} L^2\right) + 2M \left(\frac{1}{4} L^2 + \frac{2}{5} R^2\right)$$

$$= L^2 \left(\frac{m}{12} + \frac{M}{2}\right) + R^2 \frac{4M}{5} + \left(\frac{d}{2}\right)^2 \frac{m}{4} \, .$$

B. Application: trapeze with axle



Find approximate numerical values for the inertial tensors of:

B.1. The axle and its bearings

The key observation here is that *d/2* and *R* are both much less than *L* : the rod is much longer than its radius, or the radius of the sphere. Because of this, the terms proportional to R^2 or to $(d/2)^2$ are smaller the terms proportional to L^2 , by a dimensionless factor $(R/L)^2$ or $(d/2L)^2$ which is much less than 1.

The transverse moments are then approximately $L^2 \left(\frac{m}{12} + \frac{M}{2}\right)$,

and the longitudinal moment is $\frac{1}{8} m\, d^2 + \frac{4}{5} M\, R^2$.

L = 80 cm, $m = \frac{80}{83} \times 21$ g = 20 g, $M = \frac{1}{2}$(30 g - 20 g) = 5 g, $R \approx \frac{d}{2} \approx 0.4$ cm

$$I_{trans} = (80\text{ cm})^2 \left(\frac{20\text{ g}}{12} + \frac{5\text{g}}{2}\right) = 2.7\times10^4\text{ g cm}^2,$$

$$I_{long} = \left(\frac{20\text{ g}}{2} + \frac{4\times5\text{ g}}{5}\right)(0.4\text{ cm})^2 = 2.2\text{ g cm}^2 << I_{trans}\ \surd$$

$$I = \begin{pmatrix} 2.7\times10^4 & 0 & 0 \\ 0 & 2.7\times10^4 & 0 \\ 0 & 0 & 2.2 \end{pmatrix}\text{g cm}^2 = \begin{pmatrix} 2.7\times10^{-3} & 0 & 0 \\ 0 & 2.7\times10^{-3} & 0 \\ 0 & 0 & 2.2\times10^{-7} \end{pmatrix}\text{kg m}^2$$

B.2 The upper crossbar of the trapeze with the strings wound onto its ends.

The same result holds as for **B.1.**

L = 83 cm, m = 21 g, M = 0.5 g, $R \approx$ 0.4 cm

$$I_{trans} = (83\text{ cm})^2\left(\frac{21\text{ g}}{12}+\frac{0.5\text{g}}{2}\right) = 1.4\times10^4\text{ g cm}^2,$$

$$I_{long} = \left(\frac{21\text{ g}}{2}+\frac{4\times0.5\text{ g}}{5}\right)(0.4\text{ cm})^2 = 1.7\text{ g cm}^2 << I_{trans}\ \surd$$

$$I = \begin{pmatrix} 1.4\times10^4 & 0 & 0 \\ 0 & 1.4\times10^4 & 0 \\ 0 & 0 & 1.7 \end{pmatrix}\text{g cm}^2 = \begin{pmatrix} 1.4\times10^{-3} & 0 & 0 \\ 0 & 1.4\times10^{-3} & 0 \\ 0 & 0 & 1.7\times10^{-7} \end{pmatrix}\text{kg m}^2$$