1. Because U is a unitary transformation

$$U^\dagger U = I$$

Using the infinitesimal form gives

$$\begin{aligned} U^\dagger U &= I \\ \left(I - \frac{i\varepsilon}{\hbar}G\right)^\dagger \left(I - \frac{i\varepsilon}{\hbar}G\right) &= I \\ \left(I + \frac{i\varepsilon}{\hbar}G^\dagger\right)\left(I - \frac{i\varepsilon}{\hbar}G\right) &= I \\ I - \frac{i\varepsilon}{\hbar}G + \frac{i\varepsilon}{\hbar}G^\dagger + \frac{\varepsilon^2}{\hbar^2}G^\dagger G &= I \end{aligned}$$

Because the infinitesimal form is only first order, we neglect the term of second order in the small quantity ε to get

$$\begin{aligned} I - \frac{i\varepsilon}{\hbar}G + \frac{i\varepsilon}{\hbar}G^\dagger &= I \\ \frac{i\varepsilon}{\hbar}\left(G^\dagger - G\right) &= 0 \\ \Rightarrow G &= G^\dagger \end{aligned}$$

which means that G is Hermitian.

2. (a) The possible results of a measurement of the spin component S_x are always $\pm\hbar/2$ for a spin-½ particle. The probabilities are

$$\mathcal{P}_{+x} = \left|{}_x\langle +|\psi(0)\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| + \tfrac{1}{\sqrt{2}}\langle -|\right)|+\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

$$\mathcal{P}_{-x} = \left|{}_x\langle -|\psi(0)\rangle\right|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\langle +| - \tfrac{1}{\sqrt{2}}\langle -|\right)|+\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

(b) In a field aligned along the y-axis, the Hamiltonian is

$$H = -\boldsymbol{\mu}\bullet\mathbf{B} = -(\gamma\mathbf{S})\bullet B_0\hat{\mathbf{y}} = -\gamma B_0 S_y = -\frac{\gamma B_0\hbar}{2}\sigma_y$$

where $\gamma = -e/mc$. Hence the time evolution operator is

$$U(t) = e^{-iHt/\hbar} = e^{i\gamma B_0 t\sigma_y/2} = e^{i\omega_0 t\sigma_y/2}$$

where $\omega_0 = \gamma B_0 < 0$. This looks like a rotation about y by $\theta = -\omega_0 t/2$:

$$U(t) = e^{i\omega_0 t\sigma_y/2} = \cos\left(\frac{\omega_0 t}{2}\right)\mathbf{I} + i\sin\left(\frac{\omega_0 t}{2}\right)\sigma_y$$

In matrix form, we have

$$U(t) \doteq \begin{pmatrix} \cos\left(\frac{\omega_0 t}{2}\right) & \sin\left(\frac{\omega_0 t}{2}\right) \\ -\sin\left(\frac{\omega_0 t}{2}\right) & \cos\left(\frac{\omega_0 t}{2}\right) \end{pmatrix}$$

The initial state vector is

$$|\psi(0)\rangle = |+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The time-evolved state is

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \doteq \begin{pmatrix} \cos\left(\frac{\omega_0 t}{2}\right) & \sin\left(\frac{\omega_0 t}{2}\right) \\ -\sin\left(\frac{\omega_0 t}{2}\right) & \cos\left(\frac{\omega_0 t}{2}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\omega_0 t}{2}\right) \\ -\sin\left(\frac{\omega_0 t}{2}\right) \end{pmatrix}$$

The probability of measuring S_x to be $+\hbar/2$ is

$$\begin{aligned} \mathcal{P}_{+x} &= \left|{}_x\langle +|\psi(t)\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \cos\left(\frac{\omega_0 t}{2}\right) \\ -\sin\left(\frac{\omega_0 t}{2}\right) \end{pmatrix}\right|^2 \\ &= \left|\tfrac{1}{\sqrt{2}}\cos(\omega_0 t/2) - \tfrac{1}{\sqrt{2}}\sin(\omega_0 t/2)\right|^2 = \tfrac{1}{2}\left(1 - 2\cos(\omega_0 t/2)\sin(\omega_0 t/2)\right) \\ &= \tfrac{1}{2}\left(1 - \sin\omega_0 t\right) \end{aligned}$$

Note that $\omega_0 = \gamma B_0 < 0$.

3. The eigenstates of L_z are

$$|m\rangle \doteq \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

It is useful to write the state vector in terms of these eigenstates, giving

$$\begin{aligned} \psi(\rho,\phi) &= A e^{-\rho^2/2\Delta^2}\cos^3\phi = A e^{-\rho^2/2\Delta^2}\left(\frac{e^{i\phi} + e^{-i\phi}}{2}\right)^3 \\ &= \frac{A}{8} e^{-\rho^2/2\Delta^2}\left(e^{3i\phi} + 3e^{i\phi} + 3e^{-i\phi} + e^{-3i\phi}\right) \\ &\doteq \frac{A}{8} e^{-\rho^2/2\Delta^2}\sqrt{2\pi}\left(|3\rangle + 3|1\rangle + 3|-1\rangle + |-3\rangle\right) \end{aligned}$$

To find the probability of measuring L_z we project the state vector onto the L_z eigenstate in question, square the amplitude, and then sum over all possible ways to obtain that probability. We solve this problem by using the continuous radial coordinate basis $|\rho\rangle$

and integrating over all possible values of that eigenvalue (see Eqn. 12.5.38 for 3D example):

$$\mathcal{P}_{L_z=m\hbar} = \int_0^\infty \left|\langle \rho m|\psi\rangle\right|^2 \rho\, d\rho$$

Note that the radial integral is outside the absolute value to add up all the possible probabilities ($\int |c_m(\rho)|^2 \rho\, d\rho$). For the state vector given above, this results in

$$\begin{aligned}
\mathcal{P}_{L_z=m\hbar} &= \int_0^\infty \left|\langle m|\frac{A}{8}e^{-\rho^2/2\Delta^2}\sqrt{2\pi}\left(|3\rangle + 3|1\rangle + 3|-1\rangle + |-3\rangle\right)\right|^2 \rho\, d\rho \\
&= \frac{2\pi|A|^2}{8}\int_0^\infty e^{-\rho^2/\Delta^2}\left|\langle m|\left(|3\rangle + 3|1\rangle + 3|-1\rangle + |-3\rangle\right)\right|^2 \rho\, d\rho \\
&= \left\{\frac{2\pi|A|^2}{8}\int_0^\infty e^{-\rho^2/\Delta^2}\rho\, d\rho\right\}\left|\langle m|\left(|3\rangle + 3|1\rangle + 3|-1\rangle + |-3\rangle\right)\right|^2 \\
&= \left\{\frac{2\pi|A|^2}{8}\int_0^\infty e^{-\rho^2/\Delta^2}\rho\, d\rho\right\}\left|\delta_{m3} + 3\delta_{m1} + 3\delta_{m,-1} + \delta_{m,-3}\right|^2 \\
&= \left\{\frac{2\pi|A|^2}{8}\int_0^\infty e^{-\rho^2/\Delta^2}\rho\, d\rho\right\}\left\{\delta_{m3} + 9\delta_{m1} + 9\delta_{m,-1} + \delta_{m,-3}\right\}
\end{aligned}$$

By inspection, there are four possible values of the quantum number *m*: 3, 1, -1, -3. If we define the term in the first bracket as *C*, we get

$$\mathcal{P}_{L_z=3\hbar} = C$$

For $m = 1$, we get

$$\mathcal{P}_{L_z=\hbar} = 9C$$

For $m = -1$, we get

$$\mathcal{P}_{L_z=-\hbar} = 9C$$

For $m = -3$, we get

$$\mathcal{P}_{L_z=-3\hbar} = C$$

The four probabilities must sum to one, so we get

$$
\begin{aligned}
1 &= \mathcal{P}_{L_z=3h} + \mathcal{P}_{L_z=h} + \mathcal{P}_{L_z=-1h} + \mathcal{P}_{L_z=-3h} \\
&= C\left\{1+9+9+1\right\} = 20C \\
&\Rightarrow C = \frac{1}{20} \\
&\Rightarrow \mathcal{P}_{L_z=3h} = \mathcal{P}_{L_z=-3h} = \frac{1}{20} \\
&\Rightarrow \mathcal{P}_{L_z=h} + \mathcal{P}_{L_z=-1h} = \frac{9}{20}
\end{aligned}
$$

4. For distinguishable particles, we must count all possible ways of placing the three particles in one of the three states, with no restrictions. The allowed states are

$$
\begin{aligned}
&\left|aaa\right\rangle, \left|bbb\right\rangle, \left|ccc\right\rangle \\
&\left|aab\right\rangle, \left|aba\right\rangle, \left|baa\right\rangle \\
&\left|aac\right\rangle, \left|aca\right\rangle, \left|caa\right\rangle \\
&\left|bba\right\rangle, \left|bab\right\rangle, \left|abb\right\rangle \\
&\left|bbc\right\rangle, \left|bcb\right\rangle, \left|cbb\right\rangle \\
&\left|cca\right\rangle, \left|cac\right\rangle, \left|acc\right\rangle \\
&\left|ccb\right\rangle, \left|cbc\right\rangle, \left|bcc\right\rangle \\
&\left|abc\right\rangle, \left|acb\right\rangle, \left|cab\right\rangle, \left|cba\right\rangle, \left|bac\right\rangle, \left|bca\right\rangle
\end{aligned}
$$

which makes $27 = 3^3$ states.

For bosons, the states must be symmetric under interchange of any two particle labels. Apply the three-particle symmetrizer to the above states to get

$$
\begin{aligned}
\left|aaa,S\right\rangle &= \tfrac{1}{3!}\left(P_{123} + P_{132} + P_{213} + P_{231} + P_{312} + P_{321}\right)\left|aaa\right\rangle \\
&= \tfrac{1}{6}\left(\left|aaa\right\rangle + \left|aaa\right\rangle + \left|aaa\right\rangle + \left|aaa\right\rangle + \left|aaa\right\rangle + \left|aaa\right\rangle\right) \\
&\Rightarrow \left|aaa,S\right\rangle = \left|aaa\right\rangle
\end{aligned}
$$

The results of this are

$$
\begin{aligned}
&\left|aaa,S\right\rangle = \left|aaa\right\rangle \\
&\left|bbb,S\right\rangle = \left|bbb\right\rangle \\
&\left|ccc,S\right\rangle = \left|ccc\right\rangle \\
&\left|aab,S\right\rangle = \tfrac{1}{\sqrt{3}}\left(\left|aab\right\rangle + \left|aba\right\rangle + \left|baa\right\rangle\right) \\
&\left|aac,S\right\rangle = \tfrac{1}{\sqrt{3}}\left(\left|aac\right\rangle + \left|aca\right\rangle + \left|caa\right\rangle\right) \\
&\left|abb,S\right\rangle = \tfrac{1}{\sqrt{3}}\left(\left|abb\right\rangle + \left|bab\right\rangle + \left|bba\right\rangle\right) \\
&\left|bbc,S\right\rangle = \tfrac{1}{\sqrt{3}}\left(\left|bbc\right\rangle + \left|bcb\right\rangle + \left|cbb\right\rangle\right) \\
&\left|acc,S\right\rangle = \tfrac{1}{\sqrt{3}}\left(\left|acc\right\rangle + \left|cac\right\rangle + \left|cca\right\rangle\right) \\
&\left|bcc,S\right\rangle = \tfrac{1}{\sqrt{3}}\left(\left|bcc\right\rangle + \left|cbc\right\rangle + \left|ccb\right\rangle\right) \\
&\left|abc,S\right\rangle = \tfrac{1}{\sqrt{6}}\left(\left|abc\right\rangle + \left|acb\right\rangle + \left|cab\right\rangle + \left|cba\right\rangle + \left|bac\right\rangle + \left|bca\right\rangle\right)
\end{aligned}
$$

which makes 10 states.

For fermions, the states must be antisymmetric under interchange of any two particle labels. Apply the three-particle antisymmetrizer to the above states to get

$$
\begin{aligned}
\left|aaa,A\right\rangle &= \tfrac{1}{3!}\left(P_{123} - P_{132} + P_{231} - P_{213} + P_{312} - P_{321}\right)\left|aaa\right\rangle \\
&= \tfrac{1}{6}\left(\left|aaa\right\rangle - \left|aaa\right\rangle + \left|aaa\right\rangle - \left|aaa\right\rangle + \left|aaa\right\rangle - \left|aaa\right\rangle\right) \\
&\Rightarrow \left|aaa,S\right\rangle = 0
\end{aligned}
$$

For fermions, only one state is not a null vector:

$$
\left|abc,A\right\rangle = \tfrac{1}{\sqrt{6}}\left(\left|abc\right\rangle - \left|acb\right\rangle + \left|bca\right\rangle - \left|bac\right\rangle + \left|cab\right\rangle - \left|cba\right\rangle\right)
$$

which makes 1 state.