1. The bound states in the WKB approximation are found by integrating the momentum between the classical turning points:

$$\int_{x_1}^{x_2} p(x)dx = \left(n+\tfrac{1}{2}\right)\pi\hbar$$

However, the factor of ½ comes from assuming that the potential well is linear near the turning point. For the square well, that assumption is not valid, and the analysis says to not use the ½ factor. For this well, the turning points are $\pm L/2$ and the momentum is

$$E = \frac{p^2}{2m} + V(x) \Rightarrow p(x) = \sqrt{2m\left(E - V(x)\right)} = \begin{cases} \sqrt{2mE} & -L/2 < x < 0 \\ \sqrt{2m\left(E - V_0\right)} & 0 < x < L/2 \end{cases}$$

Applying the modified WKB condition gives

$$\begin{aligned} n\pi\hbar &= \int_{-L/2}^{L/2} p(x)dx = \int_{-L/2}^{0} \sqrt{2mE}\,dx + \int_{0}^{L/2} \sqrt{2m\left(E-V_0\right)}\,dx \\ &= L/2\sqrt{2mE} + L/2\sqrt{2m\left(E-V_0\right)} \\ &= L/2\sqrt{2m}\left(\sqrt{E} + \sqrt{E-V_0}\right) \end{aligned}$$

Rearrange and square

$$\begin{aligned} \sqrt{E} + \sqrt{E-V_0} &= n\pi\hbar \frac{2}{L\sqrt{2m}} \\ E + 2\sqrt{E}\sqrt{E-V_0} + E - V_0 &= 4\frac{n^2\pi^2\hbar^2}{2mL^2} \end{aligned}$$

Note that

$$E_n^0 = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

Now rearrange and square again

$$\begin{aligned} 2\sqrt{E}\sqrt{E-V_0} &= V_0 - 2E + 4E_n^0 \\ 4E^2 - 4EV_0 &= V_0^2 - 4EV_0 + 8E_n^0V_0 + 4E^2 - 16EE_n^0 + 16\left(E_n^0\right)^2 \\ 16EE_n^0 &= V_0^2 + 8E_n^0V_0 + 16\left(E_n^0\right)^2 \\ E &= E_n^0 + \frac{V_0}{2} + \frac{V_0^2}{16E_n^0} \end{aligned}$$

The first-order energy correction in perturbation theory is

$$E_n^{(1)} = \left\langle \psi_n^0 \middle| H^1 \middle| \psi_n^0 \right\rangle = \int_{-\infty}^{\infty} \psi_n^{0*}(x)\; H^1(x)\; \psi_n^0(x)dx$$

The perturbation is different in the two halves of the well, so we break the integral into two pieces, with the perturbation Hamiltonian equal to zero in the left half and V_0 in the right half:

$$E_n^{(1)} = \int_{-L/2}^{0} \psi_n^{0*}(x)\; 0\; \psi_n^0(x)dx + \int_0^{L/2} \psi_n^{0*}(x)\; V_0\; \psi_n^0(x)dx$$
$$= V_0 \int_0^{L/2} \left|\psi_n^0(x)\right|^2 dx$$

The remaining spatial integral is the integral of the probability density over the right half of the well. All the energy eigenstate probability densities are symmetric about the middle of well, so the integral is 1/2, yielding

$$E_n^{(1)} = \frac{V_0}{2}$$

for all states. So, the two methods agree to this order.

2. a) For the unperturbed case ($\varepsilon = 0$) we have

$$H_0 \doteq V_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

with eigenvalues $E_1 = V_0, E_2 = V_0, E_3 = 4V_0$ and eigenvectors

$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that $|1\rangle$ and $|2\rangle$ are degenerate and $|3\rangle$ is nondegenerate.
b) Now look at the perturbation of the nondegenerate $|3\rangle$ state. First we need to write the perturbation Hamiltonian $H' = H - H_0$

$$H' \doteq V_0 \begin{pmatrix} 0 & 2\varepsilon & 0 \\ 2\varepsilon & 0 & 3\varepsilon \\ 0 & 3\varepsilon & 0 \end{pmatrix}$$

The first-order energy correction is

$$E_n^{(1)} = \left\langle n^{(0)} \middle| H' \middle| n^{(0)} \right\rangle$$
$$E_3^{(1)} = \left\langle 3^{(0)} \middle| H' \middle| 3^{(0)} \right\rangle = 0$$

$$\boxed{E_3^{(1)} = 0}$$

The second-order energy correction is

$$E_3^{(2)} = \sum_{k \neq 3} \frac{\left|\left\langle 3^{(0)} \middle| H' \middle| k^{(0)} \right\rangle\right|^2}{E_3^{(0)} - E_k^{(0)}} = \frac{\left|\left\langle 3^{(0)} \middle| H' \middle| 1^{(0)} \right\rangle\right|^2}{E_3^{(0)} - E_1^{(0)}} + \frac{\left|\left\langle 3^{(0)} \middle| H' \middle| 2^{(0)} \right\rangle\right|^2}{E_3^{(0)} - E_2^{(0)}}$$

$$= \frac{|0|^2}{4V_0 - V_0} + \frac{|3\varepsilon V_0|^2}{4V_0 - V_0} = 3\varepsilon^2 V_0$$

Hence the corrected energy is

$$E_3 = E_3^{(0)} + E_3^{(1)} + E_3^{(2)} = 4V_0 + 0 + 3\varepsilon^2 V_0$$

$$\boxed{E_3 = V_0\left[4 + 3\varepsilon^2\right]}$$

c) Now look at the perturbation of the degenerate $|1\rangle$ and $|2\rangle$ states. Here we need to diagonalize the perturbation Hamiltonian within that 2x2 space:

$$H' \doteq V_0 \begin{pmatrix} 0 & 2\varepsilon & 0 \\ 2\varepsilon & 0 & 3\varepsilon \\ 0 & 3\varepsilon & 0 \end{pmatrix} \Rightarrow H'_{1,2} \doteq V_0 \begin{pmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{pmatrix}$$

Diagonalizing gives

$$\begin{vmatrix} -\lambda & 2\varepsilon V_0 \\ 2\varepsilon V_0 & -\lambda \end{vmatrix} = 0$$

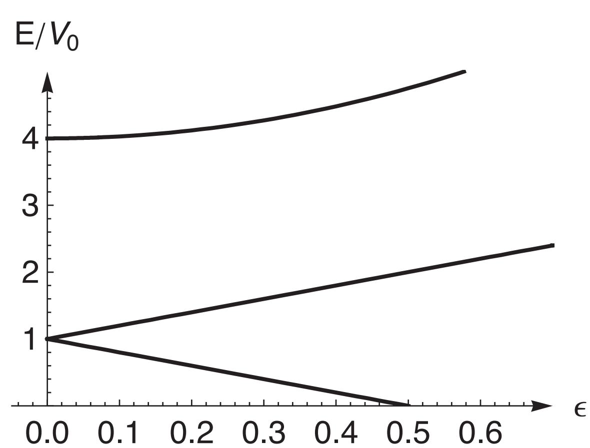
$$(-\lambda)(-\lambda) - (2\varepsilon V_0)^2 = 0$$

$$\left(\lambda^2 - 4\varepsilon^2 V_0^2\right) = 0$$

$$\lambda = \pm\, 2\varepsilon V_0$$

$$\boxed{\begin{aligned} E_1 &= E_1^{(0)} + E_1^{(1)} = V_0 + 2\varepsilon V_0 = V_0(1 + 2\varepsilon) \\ E_2 &= E_2^{(0)} + E_2^{(1)} = V_0 - 2\varepsilon V_0 = V_0(1 - 2\varepsilon) \end{aligned}}$$

d) The degenerate levels split linearly, while the nondegenerate level has a quadratic dependence and is repelled by the one lower level it is coupled to, as expected.



3. Particle #1 has spin 1 (s_1 = 1) and particle #2 has spin 1/2 (s_2 = 1/2). Use the Clebsch-Gordan coefficients to find the coupled states in terms of the uncoupled states. The state with total spin 3/2 and z-component $-\hbar/2$ is in the third column in the table:

$$|\psi\rangle = |\tfrac{3}{2}\tfrac{-1}{2}\rangle = \sqrt{\tfrac{2}{3}}|1\tfrac{1}{2},0\tfrac{-1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1\tfrac{1}{2},-1\tfrac{1}{2}\rangle$$

The possible measurements of the z-component of the spin of particle 1 are $1\hbar, 0\hbar, -1\hbar$, which correspond to $m_1 = +1, 0, -1$. To find the probability of any one result we must sum over all the possible results of the spin component of particle 2, which are $\hbar/2$, $-\hbar/2$ (corresponding to $m_2 = +1/2, -1/2$). Thus we get

$$\begin{aligned}
\mathcal{P}_{m_1=+1} &= \sum_{m_2=-1/2}^{1/2} \mathcal{P}_{m_1=+1,m_2} = \sum_{m_2=-1/2}^{1/2} \left|\langle s_1=1, s_2=\tfrac{1}{2}, m_1=+1, m_2|\psi\rangle\right|^2 \\
&= \sum_{m_2=-1/2}^{1/2} \left|\langle 1\tfrac{1}{2}1m_2|\psi\rangle\right|^2 = \sum_{m_2=-1/2}^{1/2} \left|\langle 1\tfrac{1}{2}1m_2|\left(\sqrt{\tfrac{2}{3}}|1\tfrac{1}{2},0\tfrac{-1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1\tfrac{1}{2},-1\tfrac{1}{2}\rangle\right)\right|^2 = 0 \\
\mathcal{P}_{m_1=0} &= \sum_{m_2=-1/2}^{1/2} \mathcal{P}_{m_1=0,m_2} = \sum_{m_2=-1/2}^{1/2} \left|\langle 1\tfrac{1}{2}0m_2|\psi\rangle\right|^2 \\
&= \sum_{m_2=-1/2}^{1/2} \left|\langle 1\tfrac{1}{2}0m_2|\left(\sqrt{\tfrac{2}{3}}|1\tfrac{1}{2},0\tfrac{-1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1\tfrac{1}{2},-1\tfrac{1}{2}\rangle\right)\right|^2 = \tfrac{2}{3} \\
\mathcal{P}_{m_1=-1} &= \sum_{m_2=-1/2}^{1/2} \mathcal{P}_{m_1=-1,m_2} = \sum_{m_2=-1/2}^{1/2} \left|\langle 1\tfrac{1}{2},-1m_2|\psi\rangle\right|^2 \\
&= \sum_{m_2=-1/2}^{1/2} \left|\langle 1\tfrac{1}{2},-1m_2|\left(\sqrt{\tfrac{2}{3}}|1\tfrac{1}{2},0\tfrac{-1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1\tfrac{1}{2},-1\tfrac{1}{2}\rangle\right)\right|^2 = \tfrac{1}{3}
\end{aligned}$$

The three probabilities add to unity, as they must. The probabilities of the measurement of the spin component of particle 2 are

$$\begin{aligned}
\mathcal{P}_{m_2=+1/2} &= \sum_{m_1=-1}^{1} \mathcal{P}_{m_1,m_2=+1/2} = \sum_{m_1=-1}^{1} \left|\langle s_1=1, s_2=\tfrac{1}{2}, m_1, m_2=+\tfrac{1}{2}, |\psi\rangle\right|^2 \\
&= \sum_{m_1=-1}^{1} \left|\langle 1\tfrac{1}{2}m_1\tfrac{1}{2}|\psi\rangle\right|^2 = \sum_{m_1=-1}^{1} \left|\langle 1\tfrac{1}{2}m_1\tfrac{1}{2}|\left(\sqrt{\tfrac{2}{3}}|1\tfrac{1}{2},0\tfrac{-1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1\tfrac{1}{2},-1\tfrac{1}{2}\rangle\right)\right|^2 = \tfrac{1}{3} \\
\mathcal{P}_{m_2=-1/2} &= \sum_{m_1=-1}^{1} \mathcal{P}_{m_1,m_2=-1/2} = \sum_{m_1=-1}^{1} \left|\langle 1\tfrac{1}{2}m_1\tfrac{-1}{2}|\psi\rangle\right|^2 \\
&= \sum_{m_1=-1}^{1} \left|\langle 1\tfrac{1}{2}m_1\tfrac{-1}{2}|\left(\sqrt{\tfrac{2}{3}}|1\tfrac{1}{2},0\tfrac{-1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1\tfrac{1}{2},-1\tfrac{1}{2}\rangle\right)\right|^2 = \tfrac{2}{3}
\end{aligned}$$

Again, the two probabilities add to unity, as they must.