

1 Physical constants

$$\text{fine structure constant : } \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137}$$

$$\text{Rydberg energy : } E_o = \frac{m_e e^4}{2\hbar^2 (4\pi\epsilon_0)^2} = \frac{m_e c^2 \alpha^2}{2}$$

$$\text{Bohr magneton : } \mu_B = \frac{e\hbar}{2m_e}$$

$$\text{Bohr radius : } a_o = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$$

2 Vector calculus relationships

Triple products:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

Product rules:

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \cdot (\phi \mathbf{A}) = \phi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \phi$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) + \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

Second derivatives:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Green's theorem:

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi) dV = \oint_S (\psi \nabla \phi - \phi \nabla \psi) \cdot \mathbf{dS}$$

Spherical coordinates:

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right) \hat{\mathbf{r}} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right) \hat{\boldsymbol{\theta}}$$

$$+ \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

Cylindrical coordinates:

$$\nabla f = \frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\boldsymbol{\rho}}$$

$$+ \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\boldsymbol{\phi}}$$

$$+ \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right) \hat{\mathbf{z}}$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

3 Quantum mechanics

Raising and lowering operators for angular momentum:

$$J_\pm |j, m\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle$$

Perturbation theory for nondegenerate states:

$$E_n \approx E_n^0 + \langle n | V | n \rangle + \sum_{m \neq n} \frac{|\langle n | V | m \rangle|^2}{E_n^0 - E_m^0} + \dots$$

Harmonic oscillator: $[a, a^\dagger] = 1$

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\omega\hbar}}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\omega\hbar}}$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

4 Electromagnetism

Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

Magnetic dipole field:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{r^3}$$

Energy density: $U = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$

Poynting vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

General solutions of Laplace's equation

in cylindrical coordinates (independent of z):

$$\Phi(\rho, \phi) = a_0 \log(\rho) + \sum_{n=1}^{\infty} \left(\frac{a_n}{\rho^n} + b_n \rho^n \right) (c_n \cos n\phi + d_n \sin n\phi)$$

in spherical coordinates:

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \phi)$$

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

(with azimuthal symmetry)

5 Useful math formulas

$$e^{ikr \cos \theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta)$$

$$\int_{-\infty}^{\infty} e^{ixy} dy = 2\pi \delta(x)$$

$$\int_0^{\infty} x^n e^{-x} dx = n!, \text{ integer } n$$

$$(1+x)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k$$

$$\log(n!) \approx \frac{1}{2} \log(2\pi n) + n \log(n) - n$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

$$\frac{1}{|\mathbf{x} - r'\hat{\mathbf{z}}|} = \sum_l \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos \theta)$$

Spherical Bessel functions:

$$j_0(z) = \frac{\sin z}{z} \quad n_0(z) = -\frac{\cos z}{z}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \quad n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$

Legendre polynomials:

$$P_0(x) = 1 \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_1(x) = x \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m P_l}{dx^m}$$

Spherical harmonics:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \quad Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{i2\phi}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \quad Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta \quad Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$