Using Inter-related Activities to Develop Student Understanding of Electric and Magnetic Fields

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Chapter 1

The Structure of the Overview, Instructor's Guide, and Solutions

This project attempts to make the instructional activities used during the first two Paradigms physics courses available to physics instructors at other institutions via the worldwide web. The Paradigms in Physics program at Oregon State University reorganized the sequence and structure of upper division courses to align with the ways in which professional physicists understand the field. In addition, the courses utilize reformed-based instruction that encourages students to take responsibility for their own learning. By creating instructor's guides to the activities used during the Paradigms courses, the hope is that instructors at other institutions will have the information they need to improve the instruction and content in their own courses.

Five activities were chosen as the core activities that this project addresses. The first activity involves finding the potential on the x and y axes due to two point charges on the x axis and representing that with a power series expansion. The next four activities all involve a charged ring of radius r and charge Q. The second and third activities involve finding an elliptic integral that represents the electrostatic potential and the electric field in all space. The fourth and fifth activities involve a rotating ring of charge with period T and finding an elliptic integral that represents the magnetic vector potential and magnetic field in all space.

The structure chosen to communicate the essential features of each of the five activities consists of 1) an overview, consisting of a few paragraphs, 2) a linked solution to the problems and 3) an instructor's guide which includes a guide to the key instructional aspects, including a description of the places where students face challenges and recommendations for how to help them through those challenges. This chapter will describe how and why this particular format arose and also explain the overviews, instructor's guides and solutions in more detail.

The purpose of the designed web pages is to give instructors easy access to the information they need in order to successfully use the sequence of five activities. One of the challenges was to give instructors as much as they need without giving them too much. Capturing the essential features of each activity in sufficient detail and sufficient brevity was a task that required over one hundred hours of discussions between Corinne and me.

These discussions focused both upon what was most important about the activities and how best to present them. Numerous drafts of the documents were created before a final format was agreed upon. At each stage we asked ourselves the questions:

1. What is most important about this activity?

2. What would get another instructor interested in this?

3. What would another instructor most want and need to know in order to do this activity?

4. How can we present the desired information in a user-friendly way?

The Overview

The one-page overview is designed to get the instructor interested and highlight what is most important about the activity. We decided that the instructors would first want to know what the activity is and why they would want to spend valuable class time using this activity. The overviews have two short parts. The first section entitled *Highlights of the activity* has one to three sentences describing the problem that students must solve. The second section entitled *Reasons to spend class time on this activity* has one to three short paragraphs describing the things students need to learn and how this activity addresses this need.

In the second section of the overview we make the case that there are important understandings that students at this level typically do not yet have and then briefly explain how the particular activity will help students develop these important understandings. We briefly describe how the activity allows students to wrestle with the key physics concepts in a way that builds deeper understanding. Frequently we also point out that most students do not initially have some important mathematical or physics skills and understandings that are frequently either assumed or overlooked in most curricula.

The Instructor's Guide and Worked Solutions

If the overview is successful in gaining the interest of an instructor, then it is essential that they are given the tools necessary to use the activity successfully. Linked from the overview is an instructor's guide. The instructor's

guide includes practical details such as needed props, links to handouts, and an overview of the activity. However, the "meat" of the instructor's guide fits into three general categories; prerequisite skills, the challenges students face and how to address them, and the follow-up activities that build upon what students learn during the primary activity.

The first of these categories addresses the prerequisite skills students need to be successful at this activity. One of the most important aspects of these activities is to give students algebraically complicated problems and have them learn to break them into manageable pieces. Thus, it is essential that students have enough background to make the pieces truly manageable. We needed to clearly articulate what we believed to be the important prerequisite skills and understandings that students will bring together to be successful with these activities. These prerequisites are initially listed under the heading *This activity brings together student understanding of:* and then are described in the section entitled *Student prerequisite skills.*

The list of required understandings includes such things as electrostatic potential, the physical and geometric meaning of $\frac{1}{|\vec{r}-\vec{r}'|}$, superposition, and power series expansion. The *Student prerequisite skills* section gives a more detailed description of each of these items including required formulas and yet-to-be-completed links to each of the activities Dr. Manogue uses to help students build this prerequisite knowledge.

The second main category addresses how to set up the activity and the how to effectively get students to do the essential learning. A brief section entitled *Overview* describes in one paragraph how to get students started on the activity and describes what they will be doing. The longest section of the instructor's guide is entitled, *What the students will be challenged by and how to facilitate their learning.* An alternate title we had considered for this section was, *Where students get stuck and how to get them unstuck.* As the title suggests, this section describes the types of difficulties that students have and how to effectively address these difficulties. This is the section that deals with the most essential learning and pedagogical issues.

It was decided that the instructor guides would be too hard to follow, for someone unfamiliar with the activity, unless worked solutions were also given. With worked solutions available, instructors would be able to quickly see the scope and nature of the problem and then be able to follow more clearly the discussion of where students faced difficulties in reaching successful solutions. This sent us "back to the drawing board" as we tried to establish how worked solutions and the needed commentary should be presented.

There was much debate as to whether to have running commentary in the solutions or whether to have the solutions be straight forward without pedagogical commentary and instead have the instructor's guide have the commentary that refers to the solutions. We eventually decided that instructors would like to see a clean solution done in the way a physicist would solve the problem. We could then use a separate document for the instructor's guide which would refer to the solutions and point out the places students faced challenges. We also point out ways in which student solutions differed from the ways physicists normally solve problems.

There was also debate about whether to present solutions in Math ML or LaTex. The current strategy is to have all solutions as portable document format (PDF) files made with LaTex. The hope is to eventually make the documents available in multiple formats, including PDF and MathML. It was also decided that the instructor's guide, which had been written in Microsoft Word for conversion to HTML, would be rewritten in LaTex so that equations could be included in the commentary portion.

The separate solutions and instructor's guide allow us to easily describe differences between the student approach and the expert approach. For example, it can be pointed out that when the square root of a squared quantity is taken, many students will simply "cancel out" the square and square root instead of recognizing that an absolute value of the quantity is required. Or it can be discussed that something such as finding the linear current density given the period, radius and charge of a rotating ring, which seem trivial to the instructor, actually require a great deal of thinking and effort for the students.

The instructor's guide also describes how long it typically takes for students to accomplish various steps and when it is most productive to quickly offer help and when it is most productive to let students struggle in order to gain a deeper understanding of the problem.

After learning and pedagogical considerations have been discussed, the instructor's guide has a final section entitled, *Debriefing, whole-class discussion, wrap-up and follow-up.* This section describes the types of discussions after the activity that help students focus on the main points of the activity. During this time, results of different groups can be compared and contrasted, key concepts can be emphasized, connections to other things can be made, concepts can be generalized, and applications of the learned concepts can be discussed.

This final section also contains suggestions for homework assignments

and future activities that will build upon the things students learned during this activity. Although we believe that the five main activities have value on their own, we see great value in reinforcing and expanding upon the concepts learned. In this way, the activities can be more integrated into the class as opposed to having activities that seem to appear out of nowhere and then get dropped.

The framework of these web pages allows instructors to get involved in various ways. Instructors could use the pages to spark their thinking without actually adopting the specific activities. With this approach, instructors could see the types of things we do in the Paradigms course and think of ways to create similar activities in their own courses. On the other hand, instructors could choose to use one or more of the described activities. If they choose to do a described activity, the worked solutions and instructor's guide will hopefully allow instructors to efficiently determine the essential ideas and have readily available the things they need to make the activity run smoothly.

For instructor's using some of the activities from the main sequence, the links to the activities related to prerequisite knowledge and links to follow-up activities will allow instructors to adopt varying degrees of the Paradigms courses. Instructors could choose to try anywhere from one to all five of the main sequence of activities and could adopt anywhere from none to all of the activities that build to or follow from the five main activities. Using the whole sequence along with all the build-up and follow-up activities would result in a course highly similar to the Paradigms courses.

The hope is that we have met the needs of instructors interested in the Paradigms project or in reform-based physics instruction.

Chapter 2

Theoretical Perspectives Statement

The purpose of this research is to describe the central activities used in two reformed-based upper-division physics courses that are part of Oregon State University's Paradigms program and to document what the instructor, Dr. Corinne Manogue, sees as the meaningful and important aspects of these activities. This research draws upon the case study tradition and utilizes the social constructionist ontology and epistemologies.

In their historical account of the case study, Hamel, Dufour and Fortin (1993) report that Bronislaw Malinowski, considered a founder of modern anthropology, studied in as much detail as possible the behaviors, beliefs, and rituals of particular cultures, and realized that to understand that culture required an understanding of the meanings that the members of that culture assigned to their own behaviors and rituals. The approach at that time was to become a participant observer who attempted to integrate into the culture while trying not to alter it. This study utilized the basic approach of selecting a case and then using a variety of sources to build an understanding of that case. However, the researcher's role evolved from that of participant observer to one of collaborator.

Denzin (1997) and Lincoln and Guba (1994) are among those who have argued that an understanding of meaning requires interpretation and that researchers participate in meaning creation with those being studied. From this social constructionist perspective, reality is actively constructed through social interactions and knowledge consists of mutually created understandings. With this perspective as a basis, the researcher can legitimately be an advocate and facilitator who strives along with the other participants to co-create a better world through creation of meaning and understanding (Lincoln and Guba, 1994).

An example of a social constructionist theoretical framework in science education research is entitled, *Emotional issues in science teaching: A case study of a teacher's views* (Zembylas, 2004). In this case, Zembylas initially set out to study how children's knowledge was legitimated in the classroom of an elementary school teacher, Catherine, who had been honored and recognized for her science teaching. The study evolved and focused on the role of emotions after both researcher and teacher discovered a common interest in this topic. The two then worked in collaboration over a three year period to

understand the role emotion plays in teaching science.

Undergoing a similar type of transformation, this study initially started by looking at student learning and the effectiveness of instruction but evolved into a three-year collaboration between the author and Dr. Manogue to understand the essential features and meaning in the activities in her classroom. As a former high-school science teacher who used reformed based methods and as a graduate student who had taken Dr. Manogue's physics courses, I shared an interest with Dr. Manogue in advocating for an increased usage of reformed-based curriculum and instruction in upper division college physics courses. Our combined understanding of pedagogy and learning was used to build an understanding of her instructional techniques.

In addition to audio-taped discussions between Dr. Manogue and the author, course documents, video of students working, student work, and student interviews were used to understand the student viewpoint and the nature of interaction between the instructor and her students. However, the primary purpose of using these resources was to find examples of the types of interactions and student thinking that the instructional approaches were designed to generate. Thus while data that directly contradict assertions were not intentionally ignored and were sometimes used to help build understandings, the additional data were primarily used to illustrate assertions rather than evaluate them. Thus, this study is limited to describing the types of activities that are used, their meaning, and their influence on student learning that are viewed as important by both the instructor and the author.

As part of the grant requirements that led to the creation of the Paradigms program, there has already been extensive data and analysis of the program that shows that compared to the school's earlier program, there has been improved student retention, improved grades, and generally positive student attitudes toward the program (Manogue and Krane, 2003). Thus the assumption of this study is that there already exists sufficient reason to consider Dr. Manogue's instruction to be of interest to the physics education community and that there is value in describing that instruction and the meanings given to it that have led to the instructor's conviction for using the reformed-based instructional approach..

While social constructionism was used as the theoretical framework for the research methods, the primary theoretical lens for understanding the learning and instruction was that of cognitive apprenticeship (Collins, Brown and Holum, 1991). The cognitive apprenticeship model relies on learners having opportunities to see the thinking of experts and to engage in the

types of thinking that experts engage in, including using learning strategies, metacognitive strategies and heuristic strategies or 'tricks of the trade'. The teaching method includes:

1. the expert modeling thinking,

2. coaching students as they work on tasks,

3. providing scaffolding in various forms to support the student as they work through a task,

4. having students articulate their knowledge, reasoning and problem-solving process,

5. reflecting and having students and compare their problem solving strategies to those of an expert,

6. engaging in exploration in the form of having students ask their own questions and use their problem solving abilities with a minimum of support (Collins, Brown and Holum).

Dr. Manogue has chosen the cognitive apprenticeship model as appropriate for interpreting the learning experiences that she fosters in her classroom.

An example of a study using the cognitive apprenticeship framework is Darabi's (2005) case study of a graduate course in performance systems analysis. Darabi saw an absence of examples in the literature of how the cognitive apprenticeship model was actually applied to designing a learning environment. He proceeded to describe what he believed to be the key features of that learning environment.

Similarly, there is a lack of literature describing the application of a cognitive apprenticeship model to upper-division physics courses. One goal of this study is to begin filling the gap by producing a clear example of how the cognitive apprenticeship model can be applied to a set of learning activities in these particular upper-division courses. One potential limitation of using a cognitive apprenticeship model to analyze the instructor's teaching is that although much of her teaching aligns with this model, Dr. Manogue had never actually heard of this model until the second year of this study, and thus there may be important aspects of her instruction that get overlooked once this model is taken as the primary lens for looking at her classroom learning environment.

My own interest in this study has been formed by several aspects of my own background. Twelve years as a high school and middle school math and science teacher has led to an interest in student learning and instructional strategies that encourage students to learn at a deeper level and develop knowledge that will be usable in their future lives. Being a student in physics master's program that included Dr. Manogue's classes led to my having a great personal appreciation for the learning environment in these classes compared to the lecture-centered approach practiced by many other instructors. Being a PhD student in a science education program has helped me understand different learning and instructional models. Thus, I see myself as an advocate for the dissemination and expanded use of instructional techniques such as those used in these classes. I hope to advance this cause with this case study.

Chapter 3

The Rationale for Selecting These Five Activities

The five activities chosen for this project are seen as central to the first two Paradigms courses. Why are these five activities important? One answer to this question is that they cover critical mathematics and physics content by pulling together student understanding of power series, curvilinear coordinates, electric potential, electric field, linear charge densities, magnetic vector potential, and magnetic field. Another answer is that these five activities are important pieces in advancing the "hidden curriculum" of getting students closer to "thinking like a physicist," including using geometric reasoning, understanding what algebraic symbols represent physically, and breaking a complex problem into manageable pieces. This chapter will describe the challenges students face and how these activities build student understanding of the content as well as help students begin to think more like a professional physicist.

Activity 1 - Potential due to two point charges

The first activity asks students to consider two point charges on the x axis and find the power series expansion for the potential either very close or very far from the origin on either the $x-$ or $y-$axis. The four prerequisites skills listed for this activity are understandings of 1) electrostatic potential, 2) the physical and geometric meaning of $\frac{1}{|\vec{r}-\vec{r}'|}$, 3) superposition, and 4) power series expansions.

In prior physics courses students have seen potential as $V = \frac{kq}{r}$ and have probably dealt with problems involving the potential of more than one point charge. In this respect, this activity is somewhat of a "refresher" problem for the first three understandings, pulling together knowledge students had used before this course and asking them to actively bring it to bear on a new problem. However, this problem asks students to consider potentials, superposition, and position vectors at a more sophisticated level.

There is a substantial difference between what students are asked to do here and what most students have seen previously, which usually has been solving for the potential at one point on a plane due to multiple coplanar charges by using the formula $V = \frac{kq}{r}$ and simply plugging in values. These prior problems require minimal thinking about electrostatic potential, do not

require using position vectors and vector addition, and allow for applying the superposition principle by default instead of careful thought. Instead of r being a value that can be simply plugged into a formula, creating the power series for the potential along an axis requires a more sophisticated thinking using $V(\vec{r}) = \sum_{i=1}^{N} \frac{kq_i}{|\vec{r}-\vec{r}_i|}$. Yet-to-be-created links for prerequisite skills will show activities that get students to start thinking more deeply about the concepts of potential, superposition, and the geometric meaning of adding and subtracting position vectors.

The piece of the problem that requires the creation of a power series will be something few students will have ever done in a physics context. Furthermore, making a physical interpretation of a power series will be something most students have never been asked to do.

This first activity asks only for potentials on the x and y axes. Whereas a professional physicist might immediately envision a three-dimensional potential field, many students will think entirely within the confines of two-dimensions and won't consider that they are solving for a specific case with $z = 0$ of a more generalized three-dimensional situation. This point is brought up in the whole-class discussion following the activity.

Most of the challenges students face in reaching an answer to their specific case of this problem will lie within the mathematics. The single biggest challenge for students is turning an expression like $\frac{1}{D-x}$ into something that looks like $(1+z)^p$. The most challenging aspect of this is figuring out how to get a "1" and a "something small" in the problem by factoring out a D or an x, depending on their specific question. First students need to recognize that it is possible to factor out a variable, and then they will need to correctly do the algebra. Although this will probably be the biggest challenge, the students will often spend time thinking about things a professional physicist wouldn't even imagine as requiring effort, such as wondering about having a z in the power series formula, but no z variable in the problem. Other things students will need to do in this problem include:

1. Recognizing that p in the power series formula will need to be negative

2. Correctly dealing with signs when applying the power series in conjunction with the superposition principle

3. In some cases, students will need to deal with the square root of a squared quantity, which they will often attempt to handle by "canceling

out" the square and square root without recognizing the need for an absolute value.

While most groups will not encounter extended difficulties in all these areas, very few will do this without significant effort. It is important that instructors realize the need to have students be supported while they work through these types of problems, or students will face enormous difficulty in solving problems that take for granted mathematical proficiency.

Making meaning of the answers students get is a critical portion of this activity. In individual conversations and in the whole-class discussions, it will be important for students to consider what their answer tells them about the physical situation. Having groups share their answers for different cases allows them to compare and contrast different situations, including interpreting the significance of odd and even functions. Students will also be asked to consider cases with non-zero z component and to envision the potential field in three dimensions.

One additional feature of this activity is that it allows for introduction of Laurent series, which is an expansion with the variable in the denominator, in a way that makes them seem like an almost trivial extension of power series. Expanding $(1+\frac{D}{x})^{-1}$ for $\frac{D}{x} << 1$ requires students to do nothing significantly different than expanding $(1 + \frac{x}{D})^{-1}$ for $\frac{x}{D} << 1$. However, past experience has shown that if Laurent series are introduced as a separate topic, they can be very intimidating for students.

Activity 2 - Potential due to a charged ring

The second activity is finding the electrostatic potential in all space due to a ring of charge Q and radius R. This is the first problem where students are truly asked to use three dimensional geometric thinking and to solve a complex problem by bringing together several pieces in new ways. The problem builds upon the first activity, but introduces several new aspects

The first concept students need to understand is linear charge density. Given that the ring has a charge Q students will need a few minutes to realize that the charge density $\lambda = \frac{Q}{2\pi r}$. In general students come up with this on their own without help.

Students will grapple with how the linear density relates to the "chopping and adding" aspect of integration. Students frequently leave math classes understanding integration as "the area under a curve". In this activity, students

need to consider the potential due to a charged ring. Doing this requires that students imagine chopping the ring into a bunch of infinitesimal pieces at different points in space and then figure out how to sum up the contribution to the potential from each of those pieces. Thus, integrating $\frac{\lambda(\vec{r})d\vec{r}}{|\vec{r}-\vec{r}'|}$, often requires students to think of integration a way that they have not done before.

Students must use three dimensional geometric thinking to deal with the $|\vec{r}-\vec{r}'|$ piece in the denominator, including using an appropriate coordinate system to take advantage of the symmetry of the problem. Most students will choose to center the ring at the origin and use cylindrical coordinates.

This problem raises additional challenges because students need to realize that $\frac{1}{|\vec{r}-\vec{r}'|}$ cannot be evaluated by simply using $\vec{r}'$ in curvilinear coordinates. Some instructors may even miss this point if they have not carefully considered it prior to this activity. In Cartesian coordinates $\vec{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ and $\vec{r}' = x'\hat{\mathbf{x}}' + y'\hat{\mathbf{y}}' + z'\hat{\mathbf{z}}'$. Because $\hat{\mathbf{x}} = \hat{\mathbf{x}}'$, $\hat{\mathbf{y}} = \hat{\mathbf{y}}'$, and $\hat{\mathbf{z}} = \hat{\mathbf{z}}'$, then $\vec{r}-\vec{r}' = (x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}$. Similarly, in curvilinear coordinates $\vec{r} = r\hat{\mathbf{r}}$ and $\vec{r}' = r'\hat{\mathbf{r}}'$, and $\vec{r}-\vec{r}' = r\hat{\mathbf{r}} - r'\hat{\mathbf{r}}'$ However, there is a problem because $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ can be oriented in different directions at any angle. They cannot be simply added or subtracted.

Unfortunately, solving this problem entirely in rectangular coordinates from the beginning is quite cumbersome. The solution to this dilemma is the conversion from cylindrical coordinates to Cartesian coordinates and back to cylindrical coordinates, which results in:

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + 2rR\cos(\phi-\phi') + R^2 + z^2}.$$

The final component is that students need to recognize an elliptic integral and what to do when they run into one. Refer to the solution manual, Eq.8 to see the resulting elliptic integral. Most commonly students have never seen such "unsolvable" integrals in their calculus classes. There are two approaches used to dealing with the elliptic integrals. One is to use Maple to use the elliptic integral to create a visualization of the potential field. The other is to use series expansions to understand "what is going on" along the axes.

The power series and Laurent series expansions along the z axis are highly similar to those for the two point potentials, but creating the expansion for the x axis requires using the expansion prior to integration and using a more complicated expression as the "something small" in the expansion. This also helps students understand how power series can be used for creating

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solutions to otherwise unsolvable integrals. In addition, the "messiness" of adding terms in the expansion results in students having new interest and motivation for understanding how many terms are needed for a "reasonable" approximation.

Overall, this second activity of finding the potential for a ring builds upon the knowledge students gained during the first activity with point potentials. The key additional physics and mathematics concepts used during this activity are

1. linear charge density

2. curvilinear coordinates

3. using geometric reasoning in three dimensions

4. integration as "chopping and adding"

5. using power series and Laurent series prior to integration to deal with elliptic integrals.

This activity also addresses key parts of the "hidden curriculum" of helping students think like a physicist. First, it requires students to go repeatedly back and forth among physical understanding, geometric reasoning, and algebraic symbols to get students out of the mode of seeing algebraic symbols simply as formulas into which numbers are plugged. And second, the complexity of the problem forces students to try breaking the problem into manageable pieces, which goes far beyond the "pattern matching" that students often do with textbook homework and example problems.

Activity 3 - Electric field due to a charged ring

By using the same geometry, students can focus on the differences and similarities of the electric field and the electric potential instead of dealing with both a new geometry and a new type of field and seeing the problems as unrelated. Specifically, students will need to consider the vector nature of the field. The scalar field in the previous example of electrical potential requires different geometric arguments and different symmetry considerations than the electric field.

The single most challenging new piece to this problem is dealing with $\vec{r} - \vec{r}'$ in the numerator of the integrand. As mentioned earlier $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ are

not in the same direction and thus cannot be simply subtracted in cylindrical coordinates. To a professional physicist, dealing with $rr - \vec{r}\,'$ in the numerator may seem like an almost trivial extension of the previous problem, in which students needed to use $|\vec{r} - \vec{r}\,'| = (r^2 + 2rR\cos(\phi - \phi') + R^2 + z^2)^{1/2}$ in the denominator. However, the particular expression for $|\vec{r} - \vec{r}\,'|$ was created by students in an earlier homework assignment, and individual students may have received significant assistance during the assignment. Since students are allowed to use their earlier homework, many may have simply used this expression in the previous activity without careful consideration of the geometric reasoning. Thus, students may now be genuinely constructing the geometric argument for the first time. However, even for students who did the geometric reasoning during the homework, it is often the case that students write down algebraic expressions naively without thinking about the geometry and thus end up writing incorrect things.

In fact, there are some additional mathematical complexities and considerations. When considering the special cases of the r or z axes, students are forced to deal with $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ components separately in the integral. The results allow students to gain insights into the symmetry of the problem. Refer to Eqs. 8 - 14 in the solution to see this in detail.

Activity 4 - Magnetic vector potential due to a charged ring

In this problem students must deal with the concept of current density and with the concept of a vector potential. Students will often find it surprising that their intuition about electrostatic potential is not directly applicable to vector potentials.

Instructors may see it as so simple that they wouldn't consider that it would require considerable mental effort from students to find the linear current density from a ring of charge Q and radius R rotating with period T. The reality is that the concept of current density is sufficiently new and unfamiliar that students must spend time grappling with the concept in this context. The understanding gained during this problem "pays off" when students face future problems involving linear, surface and volume current densities.

In addition to determining the magnitude of the current, students will need to consider direction. This may be the first time students have had to

consider the vector nature of current beyond simply using the "right hand rule." The understandings gained here will also be needed for the fifth activity where students are required to find the magnetic field.

The concept of magnetic vector potential is new to most students. Because most people try to understand something new by comparing it with something familiar, students will often try to use intuitions about electric potential to understand magnetic vector potential. Because expressions for both potentials contain $\frac{1}{|\vec{r} - \vec{r}'|}$, students may make the assumption that both potentials represent basically the same thing with a different constant in front. This activity will force students to confront some of the important differences between a scalar potential and a vector potential.

One example of a situation where students must confront this difference is with the magnetic vector potential along the axis of the ring. Whereas the electric potential is positive for all finite values along this axis, the magnetic vector potential is always zero. For many students this result will be counterintuitive and will cause them to think more deeply about the differences between scalar potentials and vector potentials.

Activity 5 - Magnetic field due to a charged ring

This is designed as the culminating activity for this unit which allows students to connect much prior learning in a single problem. Prior to upper division physics courses, students have little experience in dealing with anything involving the synthesizing or "pulling together" of so many things simultaneously. Students need to use symmetry and geometric understanding to be able to construct the integral using the Biot-Savart Law. For many students, this will be the messiest integral they have ever had to face (see Eq. 6 for this activity). Successfully unpacking (dealing with) this integral requires that students first believe that they are capable of tackling something like this.

The primary new piece to the problem is the vector cross product in the numerator of the integrand. Although students have done vector cross products in math classes, students will need to realize that they can apply vector cross products to this context and that doing so will help make the problem more manageable. There is ample opportunity for algebraic errors while taking the cross product, including sign errors, losing track of ϕ vs ϕ', and failure to recognize the trigonometric identity $\cos \phi \cos \phi' + \sin \phi \sin \phi' = \cos(\phi - \phi')$. However, students should be encouraged to recognize that they have all the fundamental pieces to understanding this problem along with the

ability to put them together, and that they simply need to work carefully in order to obtain a correct solution.

Collective impact of the five activities

In traditional text-based courses, juniors in undergraduate physics courses rarely develop the depth of understanding needed to solve a problem such as the magnetic field in all space for a ring of current, without resorting to "pattern matching" by comparing a specific problem with a similar problem in the text. With this sequence of activities, students deal with successively more complex problems within the context of a familiar geometry, and develop the understandings needed to successfully solve a problem like this.

An additional benefit of using the same geometry for four consecutive problems is that students develop insights and understanding about the similarities and differences between electrostatic potential, electric field, magnetic vector potential and magnetic field. These differences can be lost when a new geometry is used for each new type of problem.

By the end of this series of five activities, students will have become proficient at using increasingly complicated power series and Laurent series expansions. In doing so they will have had to wrestle with "what is small" and what an expansion tells them about a physical situation. They will also have become comfortable with elliptic integrals and using both power series and Maple visualizations to help understand the results. In addition they will have repeatedly used geometric arguments and gone back and forth among physical thinking, geometric reasoning, and algebraic symbols.

Students come from a sequence of lower-division physics classes in which figuring out which formula to plug which numbers into can be a successful strategy for receiving a good grade in the course. Students frequently learn that they can be successful even if they ignore derivations and only focus on the resulting formulas for a variety of cases. To develop new habits of the mind, the old strategies need to be rendered ineffective in a context in which students are given sufficient scaffolding for them to be successful using new and unfamiliar ways of thinking.

This strategy fits the model of cognitive apprenticeship where the expert models thinking, students are coached and supported as they work through a task, and students have to articulate their knowledge, reasoning and problem-solving process. Each of these components is an important part of each of these activities.

Collectively these activities will be starting students down the road to thinking like a physicist. Students learn to unpack progressively more complicated problems into solvable pieces using geometric reasoning and mathematical tools. This moves students away from a plugging-into-formulas approach and starts building problem-solving strategies that will be far more useful to a future physicist.

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Electrostatic Potential-Discrete Charges

Keywords: Upper-division, E and M, Electrostatic Potential, Superposition, Discrete Charges

Highlights of the activity:

- In this activity, students working in small groups use the superposition principle to write the electrostatic potential due, either to two positive charges or to a dipole, separated by a distance D.

- The groups are then asked to expand this potential in a series, either on the axis or the plane of symmetry, and either close to the center of the charge distribution or far away.

- A follow-up class discussion highlights the role of symmetry in this calculation.

Reasons to spend class time on this activity:

To be successful in this activity, students need to learn to understand the physical and geometric meaning of one over (r minus r'). This activity can be thought of as a warm-up for a series of activities that focus on unpacking successively more complicated superposition integrals.

Although our students have some experience with power series from mathematics courses, they have never before had the chance of employing the common physics strategy substituting into known series by rewriting an expression in terms of dimensionless parameters. (same link as "expand this potential")

Electrostatic Potential for 2 Discrete Charges

Instructor Guide

Keywords: Upper-division, E and M, Electrostatic Potential, Symmetry, Discrete Charges

Brief overview of the activity

Students work in small groups to create power series expansions for the electrostatic potential due to two electric charges separated by a distance D.

This activity brings together student understanding of:

1. Electrostatic potential

2. The physical and geometric meaning of $\frac{1}{\vec{r}-\vec{r}'}$

3. Superposition

4. Power series expansion

Student prerequisite skills

Before starting this activity, through traditional lecture or the optional activities linked below, students need to acquire understandings of:

1. Electrostatic potential, $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}'|}$. `Link to electrostatic potential activity.`

2. The physical and geometric meaning of $\frac{1}{\vec{r}-\vec{r}'}$. `Link to position vector activity.`

3. Superposition, $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i|}$. `Link to superposition activity.`

4. Conceptual understanding of power series expansion and knowledge of the 4-10 most common power series formulas (or students should know where to find them in a reference book) including $(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + ...$ `Link to power series activities.`

Props

- Balls to represent point charges

- Voltmeter Coordinate system (e.g. with straws or Tinkertoys)

- Poster-sized whiteboards

- Markers

- Whiteboards around room. `Link to room set-up`.

The activity - Allow 50 minutes

Overview

Students should be given or have been reminded of the formula $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}'|}$ and be assigned to work in groups of three on the Electrostatic Potential - Discrete Charges Worksheet. This activity is designed for eight groups, but can be used with as few as two groups. If working with only two groups, have each group do two of the first four problems on the worksheet. If there are more groups do more examples or have each group just do one problem. Students do their work collectively with markers on a poster-sized sheet of whiteboard at their tables. `Link to worked solutions for power series expansions`.

What the students will be challenged by and how to facilitate their learning

1. Students are unlikely to start with the general case and work toward the specific as in Eq.2 and Eq. 3 in the solutions. Instead they are likely to treat this as a two-dimensional case from the start and ignore the z axis entirely and start with something like

$$V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2}} \qquad (1)$$

Although this is not problematic for obtaining a solution to this problem, it is problematic for visualizing a 3-dimensional field intersecting

with a particular plane or axis. Frequently students are initially trying to find a formula they can plug things into to get an answer, or at least are trying to only see what is needed to obtain the required solution. By the end of the wrap-up and final whole class discussions, students should at least have considered the 3-dimensional case and be seeing their case as an example of a larger picture.

2. As an intermediate step, students will create an expression such as $V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{D-x} + \frac{1}{D+x}\right)$. Each situation has a slightly different formula. The students will definitely spend several minutes thinking about this and working through it. However, because the coordinate system is set up for them, most students are successful with this part fairly quickly. Some may have trouble turning $|\vec{r} - \vec{r}'|$ into rectangular coordinates or have problems with correct signs when applying the superposition principle. If students get stuck here, help should be given fairly quickly.

3. Students will take the equation from part 1 and develop a 4th order power series expansion. About 20 minutes will be needed for this portion of the activity. Almost all students will struggle with creating the power series (actually a Laurent series in some cases, but students do not need to be familiar with this concept before the beginning of the activity. The difference between power series and Laurent series emerges naturally in the wrap-up). Although our students have some experience with power series from mathematics courses, they have never before had the chance of employing the common physics strategy substituting into known series by rewriting an expression in terms of dimensionless parameters. Depending on the exact nature of prior instruction, students may encounter different challenges.

- Note: two of the eight cases on the worksheet are trivial (the potential on the $y$axis is zero for the $+Q$ and $-Q$ situation). Once these groups have established the correct answer and can justify it, they should be directed to work on one of the other six questions.

- If students have been exposed to Taylor's theorem $f(z) = f(a) + f'(a)(z - a) + f''(a)\frac{(z-a)^2}{2!} + $ and have not been told to use a known power series expansion, they will probably first attempt to apply this basic formula to this situation. This will rapidly lead to an algebraic

mess. In general, we let students "get stuck" at this stage for about five minutes before suggesting that they try a known power series expansion. We don't tell them which one, but they rapidly rule out formulas for trigonometric functions and other functions that clearly don't apply.

- Once students are aware that $(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + ...$ is the expansion they need to be using, they still face a substantial challenge. It is not immediately obvious to them how an expression such as $\frac{1}{|x-D|}$ can be transformed to the form $(1 + z)^p$. Simply giving students the answer at this point will defeat much of the learning possibilities of this activity. Students will need some time just to recognize that $p = -1$, but they will need much more time to determine if x or D is the smaller amount and recognize that by factoring out D they can have an expression that starts looking like $(1 + z)^p$, with $z = \frac{x}{D}$ (or $\frac{D}{x}$ or) and $p = -1$. `Link to student language and conceptual problems regarding "factoring out" terms.`

  Students should not be allowed to stay entirely stuck for too long, but they must be given time to struggle with the problem in order for the learning to "stick" and be useful in future problems. Students should be given substantial time (about 20 minutes) to grapple with this portion of the problem. Some students may make algebraic errors such as they fincorrectly factoring out D or x. These should be brought to their attention quickly. Students may also have trouble dealing with the absolute value sign. Questioning strategies should be used to ensure understanding if this is the case.

- Once the correct power series expansion for each term is established, students will then need to work through the algebra to add the two power series together. Frequently students will make sign errors. Most student mistakes on this portion are careless algebra errors and help can be given liberally as needed.

Debriefing, Whole-Class Discussion, Wrap-up and Follow-up

Group sharing

Each of the eight groups should have an opportunity to present their results to the class such that everyone can see their work. If facilities permit, this is ideally done on large whiteboards around the room.

Compare and contrast

The instructor should encourage students to compare and contrast the results for the eight situations. This should include careful attention to: 1) whether the power series is odd or even and how this relates to whether the situation is symmetric or anti-symmetric 2) whether the answers "make sense" given the physical situation and what they tell you about moving in the + or - direction on the given axis

Consideration of the 3-dimensional case

Most students will have thought about this problem entirely within two dimensions. They should be asked to consider points with a non-zero z component. Envisioning the three-dimensional potential field will help students towards the types of thinking they will need to apply to future problems.

Laurent Series

Assuming that students have not yet been exposed to Laurent series, it should be brought to their attention that a "power series" with $\frac{1}{x}$ factors is called a Laurent series. We have found that by introducing Laurent series this way, students see it as no big deal and have sufficient understanding, but if introduced before this activity they are often intimidated and confused.

Suggested homework

Determine the general case for $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q}{|\vec{r}-\vec{r}'|}$ in rectangular coordinates - Answer - $V(x,y,z) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}}$

Link to equipotential surfaces activity.

Link to "Visualizing voltage" Maple worksheet.

Activity 1: Solutions for potential due to 2 point charges

All solutions will begin the electrical potential due to point charges

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|} \tag{2}$$

$\vec{r}$ denotes the position in space at which the potential is measured and $\vec{r}_i$ denotes the position of the charge. In Cartesian coordinates this becomes

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}} \tag{3}$$

Because we are considering only the x, y plane, $z = 0$ and because the two charges are on the x-axis, then $y_i, z_i = 0$, and $N = 2$

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{2} \frac{q_i}{\sqrt{(x-x_i)^2 + y^2}} \tag{4}$$

1 x-axis

This section looks at the four cases for the potential on the x-axis. since $y = 0$, then for all four cases on the x-axis,

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{2} \frac{q_i}{\sqrt{(x-x_i)^2}} \tag{5}$$

1.1   2 positive charges, $+Q$, one at D and one at $-D$, $|x| << D$

With both charges equal to $+Q$, Eq. 4 leads to

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{(x-D)^2}} + \frac{1}{\sqrt{(x+D)^2}} \right) \tag{6}$$

Because $|x| << D$, this leads to

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{D-x} + \frac{1}{D+x} \right) \tag{7}$$

Factoring out D from the denominator yields

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{D} \left(\frac{1}{1 - \frac{x}{D}} + \frac{1}{1 + \frac{x}{D}} \right) \tag{8}$$

Which can be rewritten as

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{D} \left(\left(1 - \frac{x}{D}\right)^{-1} + \left(1 + \frac{x}{D}\right)^{-1} \right) \tag{9}$$

Using the power series $(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \dots$ results in

$$\begin{aligned} V(x, y, z) = \quad & \frac{Q}{4\pi\epsilon_0} \frac{1}{D} \left(1 + \frac{x}{D} + \frac{x^2}{D^2} + \frac{x^3}{D^3} + \dots \right) \\ + \ & \frac{Q}{4\pi\epsilon_0} \frac{1}{D} \left(1 - \frac{x}{D} + \frac{x^2}{D^2} - \frac{x^3}{D^3} + \dots \right) \end{aligned} \tag{10}$$

The odd powers cancel to produce the expansion

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{2}{D} \left(1 + \frac{x^2}{D^2} + \frac{x^4}{D^4} + \dots \right) \tag{11}$$

1.2 Opposite charges, $+Q$ at $+D$, $-Q$ at $-D$, $|x| << D$

Eq. 4 now leads to the same results for Eq. 5 except for a sign change, becoming

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{(x - D)^2}} - \frac{1}{\sqrt{(x + D)^2}} \right) \tag{12}$$

Using the same procedure as in Eq.6 - 9 before, we now have

$$\begin{aligned} V(x, y, z) = \quad & \frac{Q}{4\pi\epsilon_0} \frac{1}{D} \left(1 + \frac{x}{D} + \frac{x^2}{D^2} + \frac{x^3}{D^3} + \dots \right) \\ - \ & \frac{Q}{4\pi\epsilon_0} \frac{1}{D} \left(1 - \frac{x}{D} + \frac{x^2}{D^2} - \frac{x^3}{D^3} + \dots \right) \end{aligned} \tag{13}$$

Now the even powers cancel to become

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{2}{D} \left(\frac{x}{D} + \frac{x^3}{D^3} + \frac{x^5}{D^5} + \dots \right) \tag{14}$$

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1.3   2 positive charges, $+Q$, one at D and one at $-D$, $x >> D$

Starting with Eq.5, but now with $x >> D$,

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{x-D} + \frac{1}{x+D}\right) \tag{15}$$

Factoring out x from the denominator yields

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{x}\left(\frac{1}{1-\frac{D}{x}} + \frac{1}{1+\frac{D}{x}}\right) \tag{16}$$

Using the Laurent series expansion now results in

$$\begin{aligned} V(x,y,z) = &\quad \frac{Q}{4\pi\epsilon_0}\frac{1}{x}\left(1 + \frac{D}{x} + \frac{D^2}{x^2} + \frac{D^3}{x^3} + ...\right) \\ &+ \frac{Q}{4\pi\epsilon_0}\frac{1}{x}\left(1 - \frac{D}{x} + \frac{D^2}{x^2} - \frac{D^3}{x^3} + ...\right) \end{aligned} \tag{17}$$

The odd powers of the expansion cancel to become

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{2}{x}\left(1 + \frac{D^2}{x^2} + \frac{D^4}{x^4} + ...\right) \tag{18}$$

However, it should noted that this is an odd function, and multiplying through by $\frac{1}{x}$ results in

$$V(x,y,z) = \frac{2Q}{4\pi\epsilon_0}\left(\frac{1}{x} + \frac{D^2}{x^3} + \frac{D^4}{x^5} + ...\right) \tag{19}$$

1.4   Opposite charges, $+Q$ at $+D$, $-Q$ at $-D$, $x >> D$

Changing the sign in Eq. 14 results in the even powers of the expansion cancelling and

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{2}{x}\left(\frac{D}{x} + \frac{D^3}{x^3} + \frac{D^5}{x^5} + ...\right) \tag{20}$$

Which can be rewritten as

$$V(x,y,z) = \frac{2Q}{4\pi\epsilon_0}\left(\frac{D}{x^2} + \frac{D^3}{x^4} + \frac{D^5}{x^6} + ...\right) \tag{21}$$

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2 y-axis

This section looks at the four cases for the potential on the y-axis, where we now consider that $x = 0$ and Eq. 3 becomes

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{2} \frac{q_i}{\sqrt{x_i^2 + y^2}} \tag{22}$$

Because $x_i = \pm D$

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{2} \frac{q_i}{\sqrt{D^2 + y^2}} \tag{23}$$

2.1 2 positive charges, $+Q$, one at D and one at $-D$, $|y| << D$

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{2Q}{\sqrt{D^2 + y^2}} \tag{24}$$

Factor out D from the denominator yields

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{1}{D} \frac{2Q}{\sqrt{1 + \frac{y^2}{D^2}}} \tag{25}$$

Which can be rewritten as

$$V(x, y, z) = \frac{2Q}{4\pi\epsilon_0} \frac{1}{D} \left(1 + \frac{y^2}{D^2}\right)^{\frac{-1}{2}} \tag{26}$$

Using the power series $(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \dots$ results in

$$V(x, y, z) = \frac{2Q}{4\pi\epsilon_0} \frac{1}{D} \left(1 - \frac{1}{2}\frac{y^2}{D^2} + \frac{3}{8}\frac{y^4}{D^4} + \dots\right) \tag{27}$$

2.2 Opposite charges, $+Q$ at $+D$, $-Q$ at $-D$, either $|x| << D$ or $x >> D$

Either inspection or calculation reveals that the potential is always zero on the y-axis for this case

$$V(x, y, z) = 0 \tag{28}$$

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2.3   2 positive charges, $+Q$, one at D and one at $-D$, $y >> D$

Beginning with Eq. 24, Factoring out y from the denominator yields

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{1}{y} \frac{2Q}{\sqrt{1 + \frac{D^2}{y^2}}} \tag{29}$$

Following the same method as in Eq. 25 and 26 results is the Laurent series expansion

$$V(x, y, z) = \frac{2Q}{4\pi\epsilon_0} \frac{1}{y} \left(1 - \frac{1}{2}\frac{D^2}{y^2} + \frac{3}{8}\frac{D^4}{y^4} + \dots \right) \tag{30}$$

Electrostatic Potential-Ring of Charge

Keywords: Upper-division, E and M, Electrostatic Potential, Symmetry, Ring

Highlights of the activity:

In this activity, students working in small groups to write the electrostatic potential in all space due to a charged ring.

Reasons to spend class time on this activity:

The first concept students need to understand is linear charge density. Students must grapple with the underlying concept of charge density, but also understand how this linear density relates to the "chopping and adding" aspect of integration. Students frequently leave math classes understanding integration as "the area under a curve". This activity pushes students to transform their understanding of integration to focus on "chopping and adding".

This activity also gives students the opportunity to use curvilinear coordinates and then realize that they cannot successfully integrate without transforming them into rectangular coordinates. Understanding that (r minus r') cannot be integrated by simply using "r" in curvilinear coordinates is an important realization.

The final component is that students need to recognize an elliptic integral and what to do when they run into one. Most commonly students have never seen such "unsolvable" integrals in their calculus classes.

Electrostatic Potential for Ring of Charge

Instructor Guide

Keywords: Upper-division, E and M, Electrostatic Potential, Symmetry, Ring

Brief overview of the activity

In this activity, students work in small groups to write the electrostatic potential everywhere in space due to a charged ring.

This activity brings together student understanding of:

1. Electrostatic potential

2. Spherical and cylindrical coordinates

3. Superposition

4. Integration as "chopping and adding"

5. Linear charge density

6. 3-dimensional geometric reasoning

7. Power series expansion

Student prerequisite skills

This activity is may be used as the second in a sequence, following the electrostatic potential - discrete charges activity, or may be used on its own. Students will need understandings of:

1. The prerequisites addressed in the `electrostatic potential - discrete charges activity`.

2. Spherical and cylindrical coordinates. `Link to spherical and cylindrical coordinates activity`.

3. Integration as chopping and adding. `Link to Integration activity`.

```
4. Linear charge density
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Props

- Hula hoop or other thin ring

- Balls to represent point charges

- Voltmeter

- Coordinate system (e.g. with straws or Tinkertoys)

- Poster-sized whiteboards

- markers

- whiteboards around room. `Link to room set-up`.

The activity - Allow 50 minutes.

Overview

Part I

Students should be assigned to work in groups of three and given the following instructions using the visual of a hula hoop or other large ring: "This is a ring with total charge Q and radius R. Find the electrical potential due to this ring in all space." Students do their work collectively with markers on a poster-sized sheet of whiteboard at their tables. `Link to worked solution resulting in an elliptic integral`.

Part II

Students determine the power series expansion to represent the electrostatic potential due to the charged ring along a particular axis. Link to worked solutions for power series expansions. Note: students should not be told about part II until they have completed part I.

What the students will be challenged by and how to facilitate their learning

Part I - Finding the potential everywhere in space: Creating an elliptic integral

1. The first concept students need to understand is linear charge density. Given that the ring has a charge Q students will need a few minutes to realize that the charge density $\lambda = \frac{Q}{2\pi r}$. In general students come up with this on their own without help.

2. Students will grapple with how the linear density relates to the 'chopping and adding' aspect of integration. Students frequently leave math classes understanding integration as 'the area under a curve'. This activity pushes students to transform their understanding of integration to focus on 'chopping and adding.' Students may reach a correct representation on their own in a few minutes or the instructor may assist by using a hula hoop as a prop to help students in describing the 'chopped' bits of hoop.

3. Students must use an appropriate coordinate system to take advantage of the symmetry of the problem. Students attempting to do the problem in rectangular coordinates can be given a few minutes to struggle and see the problems that arise and then should be guided to using curvilinear coordinates. Most students will choose to do this problem in cylindrical coordinates, but an interesting problem for groups who finish early is to redo the problem in spherical coordinates.

4. Putting the whole thing together requires three dimensional geometric understanding. One of the big advantages to doing this problem in class as opposed to homework is that the instructor can interact with student making 3-dimesional arguments. Either a hoop or a ring drawn on the table can be used to ask students about the potential at points in space that are outside the plane of the ring.

5. This activity also gives students the opportunity to use curvilinear coordinates and then realize that they cannot successfully integrate without transforming them into rectangular coordinates. Understanding that $\vec{r} - \vec{r}\,'$ cannot be integrated by simply using $\vec{r}$ in curvilinear coordinates is an important realization. Some instructors may even miss

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this point if they have not carefully considered it prior to this activity. Unlike linear coordinates where $x - x'$ always refers to vectors in the same direction, this is not the case for curvilinear coordinates where r and r' can be oriented in different directions at any angle. Solving this problem entirely in rectangular coordinates from the beginning is overly cumbersome, but the curvilinear coordinates which very successfully simplify the problem can lead one to incorrectly think that using $\vec{r} - \vec{r}'$ in curvilinear coordinates can be successfully integrated. To see how this fits into the whole process, see the `link to worked solution resulting in an elliptic integral`.

6. The final component is that students need to recognize an elliptic integral and what to do when they run into one. Most commonly students have never seen such 'unsolvable' integrals in their calculus classes. In our case we had students do the power series expansion before the integral (see below).

Part II - Finding the potential along an axis: Power series expansion

With the charged ring in the x, y plane, students will make the power series expansion for either near or far from the plane on the z axis or near or far from the z axis in the x, y plane. Once all students have made significant progress toward finding the integral from part I, and some students have successfully determined it, then the instructor can quickly have a whole class discussion followed by telling students to now create a power series expansion. The instructor may choose to have the whole class do one particular case or have different groups do different cases. `Link to worked solutions for power series expansions`.

If you are doing this activity without having had students first create power series expansions for the electrostatic potential due to two charges, students will probably find this portion of the activity very challenging. If they have already done the `Electrostatic Potential - Discrete Charges` activity, or similar activity, students will probably be successful with the y axis case without a lot of assistance because it is very similar to the y axis case for the two $+Q$ point charges. However, the y axis presents a new challenges because the "something small" is two terms. It will probably not be obvious for students to let $\epsilon = \frac{2R}{r}\cos\phi' + \frac{R^2}{r^2}$ ($see\,Eq.17\,in\,the\,solutions$) and sugges-

tions should be given to avoid having them stuck for a long period of time. Once this has been done, students may also have trouble combining terms of the same order. For example the ϵ^2 term results in a third and forth order term in the expansion and students may not realize that to get a valid third order expansion they need to calculate the ϵ^3 term.

Debriefing, Whole-Class Discussion, Wrap-up and Follow-up

- Discuss which variables are variable and which variables are held constant - Students frequently think of anything represented by a letter as a 'variable' and do not realize that for each particular situation certain variables remain constant during integration. For example students do frequently do not see that the R representing the radius of the ring is held constant during integrating over all space while the r representing the distance to the origin is varying. Understanding this is something trained physicists do naturally while students frequently don't even consider it. This is an important discussion that helps students understand this particular ring problem and also lays the groundwork for better understanding of integration in a variety of contexts. `Link to helping students understand what is variable are what is held constant.`

- Maple representation of elliptic integral - After finding the elliptic integral and doing the power series expansion, students can see what electric potential 'looks like' over all space by using Maple. `Link to Maple worksheet`

Suggested Homework

- Use Maple to solve a nasty integral

- Integrate over a volume with a charge distribution

Activity 2: Solution for electric potential due to a ring

Find the electrostatic potential in all space due to a ring with total charge Q and radius R

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|} \tag{31}$$

For a ring of charge this becomes

$$V(\vec{r}) = \int_{\text{ring}} \frac{1}{4\pi\epsilon_0} \frac{\lambda(\vec{r}')\,|d\vec{r}'|}{|\vec{r} - \vec{r}'|} \tag{32}$$

where $\vec{r}$ denotes the position in space at which the potential is measured and $\vec{r}'$ denotes the position of the charge.

In cylindrical coordinates, $|d\vec{r}'| = R\,d\phi'$, where R is the radius of the ring. Thus,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda(\vec{r}')\,R\,d\phi'}{|\vec{r} - \vec{r}'|} \tag{33}$$

Assuming constant linear charge density for a ring with charge Q and radius R, $\lambda(\vec{r}') = \frac{Q}{2\pi R}$ Thus,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{|\vec{r} - \vec{r}'|} \tag{34}$$

Since $\vec{r}$ and $\vec{r}'$ are not necessarily in the same direction, we cannot simply leave $|\vec{r} - \vec{r}'|$ in curvilinear coordinates and integrate directly. One solution to this problem is to rewrite $|\vec{r} - \vec{r}'|$ in cartesian coordinates

$$|\vec{r} - \vec{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \tag{35}$$

Setting the ring in the x, y plane with the center at the origin and then rewriting in cylindrical coordinates results in

$$|\vec{r} - \vec{r}'| = \sqrt{(r\cos\phi - R\cos\phi')^2 + (r\sin\phi - R\sin\phi')^2 + (z - 0)^2} \tag{36}$$

Which simplifies to

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} \tag{37}$$

Substituting into Eq. 4 results in the elliptic integral

$$V(r, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{38}$$

3   The z axis

For points on the z axis, $r = 0$ and the integral simplifies to

$$V(r, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{\sqrt{R^2 + z^2}} \tag{39}$$

And thus

$$V(r, \phi, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{R^2 + z^2}} \tag{40}$$

3.1   Power series expansions for z axis

To create the power series expansion for $|z| << R$, factor out R from the denominator

$$V(r, \phi, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{R} \frac{1}{\sqrt{1 + \frac{z^2}{R^2}}} \tag{41}$$

Using the power series $(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \dots$ results in

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{R}\left(1 - \frac{1}{2}\frac{z^2}{R^2} + \frac{3}{8}\frac{z^4}{R^4} + \dots\right) \tag{42}$$

The power series expansion for $z >> R$ is

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2} + \frac{3}{8}\frac{R^4}{z^4} + \dots\right) \tag{43}$$

4 The x axis

For points on the x axis, $z = 0$ and $\phi = 0$, so the integral simplifies to

$$V(r, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{\sqrt{r^2 - 2rR\cos\phi' + R^2}} \tag{44}$$

Which can be rewritten as

$$V(r, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} (r^2 - 2rR\cos\phi' + R^2)^{-1/2} d\phi' \tag{45}$$

In this case the power series expansion can be done before integration and then the power series can be integrated. For $x >> R$, factor out an $1/r$ to obtain

$$V(r, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{1}{r} \left(1 - \frac{2R}{r}\cos\phi' + \frac{R^2}{r^2}\right)^{-1/2} d\phi' \tag{46}$$

Let $\epsilon = \frac{2R}{r}\cos\phi' + \frac{R^2}{r^2}$

$$V(r, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \frac{1}{r} \int_0^{2\pi} (1 - \epsilon)^{-1/2} d\phi' \tag{47}$$

The power series expansion now yields

$$V(r, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \frac{1}{r} \int_0^{2\pi} \left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{15}{48}\epsilon^3 + ...\right) d\phi' \tag{48}$$

Substituting $\frac{2R}{r}\cos\phi' + \frac{R^2}{r^2}$ for ϵ results in the integrand

$$1 - \frac{1}{2}\frac{2R}{r}\cos\phi' - \frac{R^2}{r^2} + \frac{3}{8}\frac{4R^2}{r^2}\cos^2\phi' \quad + \quad \frac{3}{8}\frac{4R^3}{r^3}\cos\phi' + ...$$

$$- \frac{5}{16}\frac{8R^3}{r^3}\cos^3\phi' - ... \tag{49}$$

42

Adding like terms and doing the integral results in

$$V(r, \phi, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \frac{1}{r} \left(2\pi - \frac{\pi}{2} \frac{R^2}{r^2} + \dots \right) \tag{50}$$

Which can be simplified to

$$V(r, \phi, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \left(1 - \frac{1}{4} \frac{R^2}{r^2} + \dots \right) \tag{51}$$

Electrostatic Field–Ring of Charge-Overview

Keywords: Upper-division, E and M, Electric Field, Symmetry, Ring

Highlights of the activity:

In this activity, students working in small groups to find the electric field in all space due to a charged ring.

Reasons to spend class time on this activity:

Two key concepts are introduced through this activity, one is gradient and the other is curvilinear vectors in the integrand.

Most students have a background that leads them to think of fields in terms of forces. Starting with electric potentials and using gradients to understand electric fields helps students build a much richer and deeper understanding. In addition developing the idea of fields from potentials "levels the playing field" because students have vastly different prior knowledge of the relationship between fields and forces, whereas nearly all students are novices at making the connection between potentials and fields. In this way the class can build a common understanding of fields without some students being bored and others being overwhelmed.

Electrostatic Field for Ring of Charge

Instructor Guide

Keywords: Upper-division, E and M, Electric Field, Symmetry, Ring

Brief overview of the activity

In this activity, students work in small groups to find the electric field everywhere in space due to a charged ring.

This activity brings together student understanding of:

1. Electric Field

2. Spherical and cylindrical coordinates

3. Superposition

4. Integration as "chopping and adding"

5. Understanding of which variables are variable and which are held constant during integration

6. Linear charge density

7. 3-dimensioinal geometric reasoning

Student prerequisite skills

This activity was designed to be used following the electrostatic potential - ring activity. If this activity is being used on its own, the instructor should look at the electric potential activity to understand the types of things students will encounter with this ring problem. Students will need understandings of:

1. The prerequisites addressed in the electrostatic potential - ring activity.

2. Which variables are variable and which are held constant during integration. `Link to helping students understand what is variable are what is held constant.`

3. Electric field

Props

- Hula hoop or other thin ring

- Balls to represent point charges

- Coordinate system (e.g. with straws or Tinkertoys)

- Poster-sized whiteboards

- markers

- whiteboards around room. `Link to room set-up`.

The activity - Allow 30 minutes.

Overview

Students should be assigned to work in groups of three and given the following instructions using the visual of a hula hoop or other large ring: "This is a ring with total charge Q and radius R. Find the electrical field due to this ring in all space." Students do their work collectively with markers on a poster-sized sheet of whiteboard at their tables. `Link to worked solution resulting in an elliptic integral`.

What the students will be challenged by and how to facilitate their learning

This description assumes students have already completed the `Electrostatic Potential - Ring` activity. This activity expands upon and reinforces the concepts from the previous activity. If students thoroughly understood parts of the previous activity, they may find this activity fairly easy. On the other hand, if they received a lot of help during the previous activity and had a weaker understanding, this will help them to understand those previous concepts. The instructor can help "tweak" and extend the understanding of the strongest students and can make sure the weaker students are truly understanding the essential concepts

1. Students will need to consider the vector nature of the field. The scalar field in the previous example of electrical potential requires different geometric arguments and different symmetry considerations than the electric field. Thinking about these differences helps students more clearly understand the differences between electrical potential and electric field. Prior to this activity, we had a lecture about the vector nature of electric fields that goes beyond the $V = \frac{kq}{r^2}$ from earlier courses. `Link to helping students understand electric field vectors`.

2. As in the previous activity, this activity also gives students the opportunity to use curvilinear coordinates and rectangular coordinates in combination to create an elliptic integral. For a description of the issues involved, see the electrostatic potential - ring activity, item 5 under "what students will be challenged by and how to facilitate their learning." For a worked solution to the electric field problem see the `link to the worked solution resulting in an elliptic integral`.

Debriefing, Whole-Class Discussion, Wrap-up and Follow-up

- Discuss-

- Maple-Link to Maple worksheet

- Suggested homework -

Activity 3: Solution for electric field

Find the electric field in all space due to a ring with total charge Q and radius R

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} \tag{52}$$

For a ring of charge this becomes

$$\vec{E} = \int_{\text{ring}} \frac{1}{4\pi\epsilon_0} \frac{\lambda(\vec{r}')\,|d\vec{r}'|\,\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \tag{53}$$

where $\vec{r}$ denotes the position in space at which the electric field is measured and $\vec{r}'$ denotes the position of the charge.

In cylindrical coordinates, $|d\vec{r}'| = R\,d\phi'$, where R is the radius of the ring. Thus,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{0}^{2\pi} \frac{\lambda(\vec{r}')\,R\,d\phi'\,\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \tag{54}$$

Assuming constant linear charge density for a ring with charge Q and radius R, $\lambda(\vec{r}') = \frac{Q}{2\pi R}$ Thus,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_{0}^{2\pi} \frac{d\phi'\,\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \tag{55}$$

Since $\vec{r}$ and $\vec{r}'$ are not necessarily in the same direction, we cannot simply leave $|\vec{r} - \vec{r}'|$ in curvilinear coordinates and integrate directly. One solution to this problem is to go back and forth between cylindrical and cartesian coordinates to represent $\vec{r} - \vec{r}'$

$$\begin{aligned}
\vec{r} - \vec{r}' &= (x - x')\hat{\imath} + (y - y')\hat{\jmath} + (z - z')\hat{k} \tag{56} \\
&= (r\cos\phi - R\cos\phi')\hat{\imath} + (r\sin\phi - R\sin\phi')\hat{\jmath} + (z - z')\hat{k} \tag{57}
\end{aligned}$$

And

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} \tag{58}$$

The electric field can now be represented by the elliptic integral

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int\limits_0^{2\pi} \frac{[(r\cos\phi - R\cos\phi')\hat{\imath} + (r\sin\phi - R\sin\phi')\hat{\jmath} + z\hat{k}]\,d\phi'}{(r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2)^{3/2}} \quad (59)$$

5   The z axis

For points on the z axis, $r = 0$ and the integral simplifies to

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int\limits_0^{2\pi} \frac{[-R\cos\phi'\,\hat{\imath} + -R\sin\phi'\,\hat{\jmath} + z\hat{k}]\,d\phi'}{(R^2 + z^2)^{3/2}} \quad (60)$$

Doing the integral results in

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{z\hat{k}}{(R^2 + z^2)^{3/2}} \quad (61)$$

6   The x axis

For points on the x axis, $z = 0$ and $\phi = 0$, so the integral simplifies to

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int\limits_0^{2\pi} \frac{[(r - R\cos\phi')\,\hat{\imath} + -R\sin\phi'\,\hat{\jmath}]\,d\phi'}{(r^2 - 2rR\cos\phi' + R^2)^{3/2}} \quad (62)$$

let $u = r^2 - 2rR\cos\phi' + R^2$, then $du = 2rR\sin\phi'd\phi'$, and for the $\hat{\jmath}$ component the integral becomes

$$\vec{E}_j = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \frac{1}{2r} \int\limits_0^{2\pi} \frac{du\hat{\jmath}}{u^{3/2}} \quad (63)$$

Doing the integral results in

$$\vec{E}_j = 0 \quad (64)$$

Thus the $\hat{\jmath}$ component disappears and results in the elliptic integral with only an $\hat{\imath}$ component

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int\limits_0^{2\pi} \frac{(r - R\cos\phi')\,\hat{\imath}\,d\phi'}{(r^2 - 2rR\cos\phi' + R^2)^{3/2}} \quad (65)$$

Magnetic Vector Potential -Ring of Charge - Overview

Keywords: Upper-division, E and M, Magnetic Vector Potential, Symmetry, Ring

Highlights of the activity:

Working in small groups students are asked to consider a ring with charge Q, and radius R rotating about its axis with period T and create an integral expression for the vector potential caused by this ring everywhere in space. Students also develop the power series expansion for the potential near the center or far from the ring.

Reasons to spend class time on this activity:

- Surprisingly, it is non-trivial for students to start with Q, R, and T and determine the current density. Although the process does not take long, having students clarify these relationships is important for building understanding. For physicists the picture generated in one's mind can be make this appear so simple that is easy to overlook the important connections that students need to make regarding current density before dealing with other aspects of the problem.

- This activity allows students to build upon prior understandings and apply them specifically to vector potentials. Even though students had prior experience with this geometry, the instructor found herself discussing with several students the difference between r and r' and which variables could be held constant during integration.

Magnetic Vector Potential for Ring

Instructor's Guide

Keywords: Upper-division, E and M, Magnetic Vector Potential, Symmetry, Ring

Brief overview of the activity

In this activity, students work in small groups to write the magnetic potential everywhere in space due to a ring of charge Q and radius R spinning with period T.

This activity brings together student understanding of:

1. Electrostatic potential

2. Spherical and cylindrical coordinates

3. Superposition

4. Integration as "chopping and adding"

5. Linear charge density

6. 3-dimensional geometric reasoning

7. Power series expansion

Student prerequisite skills

This activity is may be used as the fourth in a sequence, following the electric field activity, or may be used on its own. Students will need understandings of:

1. The prerequisites addressed in the `electrostatic field activity`.

2. Spherical and cylindrical coordinates. `Link to spherical and cylindrical coordinates activity`.

3. Integration as chopping and adding. `Link to Integration activity`.

```
4. Linear charge density
```

Props

- Hula hoop or other thin ring

- Balls to represent point charges

- Voltmeter

- Coordinate system (e.g. with straws or Tinkertoys)

- Poster-sized whiteboards

- markers

- whiteboards around room. `Link to room set-up`.

The activity - Allow 50 minutes.

Overview

What the students will be challenged by and how to facilitate their learning

Highlights of the activity

Working in small groups students are asked to consider a ring with charge Q, and radius R rotating about its axis with period T and create an integral expression for the vector potential caused by this ring everywhere in space. Students also develop the power series expansion for the potential near the center or far from the ring.

Reasons to spend class time on this activity:

1. Surprisingly, it is non-trivial for students to start with Q, R, and T and determine the current density. Although the process does not take long, having students clarify these relationships is important for building understanding. For physicists the picture generated in one's mind can be make this appear so simple that is easy to overlook the

important connections that students need to make regarding current density before dealing with other aspects of the problem.

2. This activity allows students to build upon prior understandings and apply them specifically to vector potentials. Even though students had prior experience with this geometry, the instructor found herself discussing with several students the difference between r and r' and which variables could be held constant during integration.

In this problem students must deal with the concept of current density and with the concept of a vector potential. Students will often find it surprising that their intuition about electrostatic potential is not directly applicable to vector potentials.

Instructors may see it as so simple that they wouldn't consider that it would require considerable mental effort from students to find the linear current density from a ring of charge Q and radius R rotating with period T. The reality is that the concept of current density is sufficiently new and unfamiliar that students must spend time grappling with the concept in this context. The understanding gained during this problem "pays off" when students face future problems involving linear, surface and volume current densities.

In addition to determining the magnitude of the current, students will need to consider direction. This may be the first time students have had to consider the vector nature of current beyond simply using the "right hand rule." The understandings gained here will also be needed for the fifth activity where students are required to find the magnetic field.

The concept of magnetic vector potential is new to most students. Since most people try to understand something new by comparing it with something familiar, students will often try to use intuitions about electric potential to understand magnetic vector potential. Because expressions for both potentials contain $\frac{1}{|\vec{r}-\vec{r}'|}$, students may make the assumption that both potentials basically the same thing with a different constant in front. This activity will force students to confront some of the important differences between a scalar potential and a vector potential.

One example of a situation where students must confront this difference is with the magnetic vector potential along the axis of the ring. Whereas the electric potential is positive for all finite values along this axis, the magnetic vector potential is always zero. For many students this result will be coun-

terintuitive and will cause them to think more deeply about the differences between scalar potentials and vector potentials.

Solution for magnetic vector potential in all space due to a ring with total charge Q and radius R rotating with a period T

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\text{ring}} \frac{\vec{I}(\vec{r}') \, dl'}{|\vec{r} - \vec{r}'|} \tag{66}$$

where $\vec{r}$ denotes the position in space at which the magnetic vector potential is measured and $\vec{r}'$ denotes the position of the current segment.

For the current

$$\vec{I}(\vec{r}') = \lambda(\vec{r})\vec{v} = \frac{Q}{2\pi} \frac{2\pi R}{T} \hat{\phi} = \frac{QR}{T} \hat{\phi} \tag{67}$$

$$= \frac{QR}{T}(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) \tag{68}$$

In cylindrical coordinates, $dl' = R \, d\phi'$, and, as discussed in previous solutions,

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} \tag{69}$$

Thus

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{QR}{T} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})R d\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{70}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})d\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{71}$$

7   The z axis

For points on the z axis, $r = 0$ and the integral simplifies to

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})d\phi'}{\sqrt{R^2 + z^2}} \tag{72}$$

Doing the integral results in

$$\vec{A}(\vec{r}) = 0 \tag{73}$$

55

8 The x axis

For points on the x axis, $z = 0$ and $\phi = 0$, so the integral simplifies to

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})d\phi'}{\sqrt{r^2 - 2rR\cos\phi' + R^2}} \tag{74}$$

This results in a very similar situation as the case for electric field on the x axis, except that now we will address the $\hat{\boldsymbol{\imath}}$ component instead of the $\hat{\boldsymbol{\jmath}}$ component. Using the same process we let $u = x^2 - 2xR\cos\phi' + R^2$, then $du = 2xR\sin\phi'd\phi'$, and for the $\hat{\boldsymbol{\imath}}$ component the integral becomes

$$\vec{A}_x(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \frac{-1}{2x} \int_0^{2\pi} \frac{du\hat{\boldsymbol{\jmath}}}{u^{1/2}} \tag{75}$$

Doing the integral, we find

$$\vec{A}_x(\vec{r}) = 0 \tag{76}$$

Thus the $\hat{\boldsymbol{\imath}}$ component disappears and we are left with an elliptic integral with only a $\hat{\boldsymbol{\jmath}}$ component

$$\vec{A}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{\cos\phi' \,\hat{\boldsymbol{\jmath}}\, d\phi'}{\sqrt{r^2 - 2rR\cos\phi' + R^2}} \tag{77}$$

Magnetic Field– Ring of Current -Overview

Keywords: Upper-division, E and M, Magnetic Field, Symmetry, Ring

Highlights of the activity:

Working in small groups students are asked to consider a ring with charge Q, and radius R rotating about its axis with period T and create an integral expression for the magnetic field caused by this ring everywhere in space.

Reasons to spend class time on this activity:

Students need to be able to unpack (deal with) the messy integral of the Biot-Savart Law. Prior to upper division physics courses, students have little experience in dealing with anything involving the synthesizing or "pulling together" of so many things simultaneously. The use of symmetry plays a critical role in succeeding at this activity. This activity is designed as a culminating activity for this unit which allows students to connect much prior learning in a single problem.

Magnetic Field for Ring - Instructor's Guide

Keywords: Upper-division, E and M, Magnetic Field, Symmetry, Ring

Highlights of the activity

Working in small groups students are asked to consider a ring with charge Q, and radius R rotating about its axis with period T and create an integral expression for the magnetic field caused by this ring everywhere in space.

Reasons to spend class time on this activity

This is designed as the culminating activity for this unit which allows students to connect much prior learning in a single problem. Prior to upper division physics courses, students have little experience in dealing with anything involving the synthesizing or "pulling together" of so many things simultaneously. Students need to use symmetry and geometric understanding to be able to construct the integral using the Biot-Savart Law. For many students, this will be the messiest integral they have ever had to face (see Eq. 6 in the solutions for this activity). Successfully unpacking (dealing with) this integral requires that students first believe that they are capable of tackling something like this.

The primary new piece to the problem is the vector cross product in the numerator of the integrand. Although students have done vector cross products in math classes, students will need to realize that they can apply vector cross products to this context and that doing so will help make the problem more manageable. There is ample opportunity for algebraic errors while taking the cross product, including sign errors, losing track of ϕ vs ϕ', and failure to recognize the trigonometric identity $\cos\phi\cos\phi' + \sin\phi\sin\phi' = \cos(\phi - \phi')$. However, students should be encouraged to recognize that they have all the fundamental pieces to understanding this problem along with the ability to put them together, and that they simply need to work carefully in order to obtain a correct solution.

Solution for magnetic field in all space due to a ring with total charge Q and radius R rotating with a period T

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int\limits_{\text{ring}} \frac{\vec{I}(\vec{r}') \times (\vec{r} - \vec{r}') \, dl'}{|\vec{r} - \vec{r}'|^3} \tag{78}$$

where $\vec{r}$ denotes the position in space at which the magnetic field is measured and $\vec{r}'$ denotes the position of the current segment. As described in previous solutions,

$$dl' = R \, d\phi' \tag{79}$$

$$\vec{I}(\vec{r}') = \frac{QR}{T}(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) \tag{80}$$

$$\vec{r} - \vec{r}' = (r\cos\phi - R\cos\phi')\hat{\boldsymbol{\imath}} + (r\sin\phi - R\sin\phi')\hat{\boldsymbol{\jmath}} + (z - z')\hat{\boldsymbol{k}} \tag{81}$$

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} \tag{82}$$

Thus $\vec{B}(\vec{r}) =$

$$\frac{\mu_0}{4\pi}\frac{QR^2}{T} \int\limits_0^{2\pi} \frac{(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) \times [(r\cos\phi - R\cos\phi')\hat{\boldsymbol{\imath}} + (r\sin\phi - R\sin\phi')\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}]d\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{83}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR^2}{T} \int\limits_0^{2\pi} \frac{(z\sin\phi'\hat{\boldsymbol{\imath}} + z\cos\phi'\hat{\boldsymbol{\jmath}} + [R + \cos(\phi - \phi')]\hat{\mathbf{k}})d\phi'}{\sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2}} \tag{84}$$

9 The z axis

For points on the z axis, $r = 0$ and ϕ can be arbitrarily taken as zero. Thus, the integral simplifies to

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{QR^2}{T} \int\limits_0^{2\pi} \frac{[z\sin\phi'\hat{\boldsymbol{\imath}} + z\cos\phi'\hat{\boldsymbol{\jmath}} + (R + \cos\phi')\hat{\mathbf{k}}]d\phi'}{\sqrt{R^2 + z^2}} \tag{85}$$

Doing the integral results in

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \frac{2\pi R}{\sqrt{R^2 + z^2}} \tag{86}$$

10   The x axis

For points on the x axis, $z = 0$ and $\phi = 0$, so the integral simplifies to

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int_0^{2\pi} \frac{[z \sin \phi' \hat{\imath} + z \cos \phi' \hat{\jmath} + (R + \cos \phi') \hat{\mathbf{k}}] d\phi'}{\sqrt{r^2 - 2rR \cos \phi' + R^2}} \tag{87}$$

using the same process as the previous two solutions, the $\hat{\imath}$ and the component disappears and the remaining elliptic integral is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{QR^2}{T} \int_0^{2\pi} \frac{[z \cos \phi' \hat{\jmath} + (R + \cos \phi') \hat{\mathbf{k}}] d\phi'}{\sqrt{r^2 - 2rR \cos \phi' + R^2}} \tag{88}$$