Electrostatic Potential Due to a Ring of Charge (Code: 2D)

To solve this problem, I started with:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r_i}|} \tag{1}$$

This equation gives the electrostatic potential due to N point charges. In this equation, q_i represents the individual charges, $|\vec{r} - \vec{r_i}|$ is the distance between the point we are measuring the potential at ($\vec{r}$) and location of the charge ($\vec{r_i}$) and ϵ_0 is the permittivity of free space. From this equation we can see that V is directly proportional to the amount of charge, and inversely proportional to the distance between $\vec{r}$ and $\vec{r_i}$.

This equation took on a similar form for a linear charge distribution:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r'})|d\vec{r'}|}{|\vec{r} - \vec{r'}|} \tag{2}$$

The prime notation used here is a convenient way to denote variables that are related to the position of the charge. Thus, $\vec{r'}$ is the position of the piece of charge, and $|d\vec{r'}|$ is the little distance used to integrate around the ring. Since I was given the total charge and radius of the ring and told that it was a *constant* charge density, I had the following expression:

$$\lambda = \frac{Q}{2\pi R} \tag{3}$$

After plugging this into Eqn (2) I had:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R} \int \frac{|d\vec{r'}|}{|\vec{r} - \vec{r'}|} \tag{4}$$

I used cylindrical coordinates because of the geometry of the ring. In this system $|d\vec{r'}|$ becomes $Rd\phi'$ and the limits of integration then become $[0, 2\pi]$ to sum over the entire ring. Applying this to Eqn (4) yeilds:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\phi'}{|\vec{r} - \vec{r'}|} \tag{5}$$

It seemed to me that I was ready to integrate now, but because $\vec{r}$ and $\vec{r'}$ won't always point in the same direction, I needed to write them out explicitly. Using the solution from our homework assignment to write out $|\vec{r}-\vec{r'}|$'s components in cartesian form and converting them to in polar form, I had

$$V(r,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{\sqrt{(r^2+R^2+z^2-2rR\cos(\phi-\phi')}} \tag{6}$$

This is an integral that can't be solved by hand.

The next step was to set $r = 0$ and Eqn (7) became:

$$V(r\!=\!0,\phi,z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\phi'}{\sqrt{R^2+z^2}} \tag{7}$$

I integrated to get:

$$V(r\!=\!0,\phi,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2+z^2}} \tag{8}$$

After recognizing that I needed to use the power series

$$(1+c)^p = 1 + pc + \frac{p(p-1)}{2!}c^2 + ... \tag{9}$$

I factored out an R from the denominator so that $c << 1$. I then had:

$$V = \frac{Q}{4\pi\epsilon_0 R}\left(1+\frac{z^2}{R^2}\right)^{-\frac{1}{2}} \tag{10}$$

Using Eqn (10) and recognizing that $p = -\frac{1}{2}$ and $c = \frac{z^2}{R^2}$, I obtained the following:

$$V(z) = \frac{Q}{4\pi\epsilon_0 R}\left(1-\frac{z^2}{2R^2}+\frac{3z^4}{8R^4}+...\right) \tag{11}$$

I discovered that unless I focused on a specific axis, the simplest form of an expression can came as an unsolveable integral. I probably would not have recognized this at first. I also discovered that changing the position vectors into rectangular coordinates and then describing each of their rectangular components in polar form can allow for easier manipulation. After focusing on the z-axis I saw that an otherwise difficult integral to calculate can become manageable. After expanding my solution in a power series that was familiar to me, I also saw that the electrostatic potential contained only even powers of z. The group that evaluated points far from 0 along the z-axis had an answer that was similar to mine, but with the z terms in the denominator and the R terms in the numerator.