Activity 1: Solutions for potential due to 2 point charges

All solutions will begin the electrical potential due to point charges

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i|} \tag{1}$$

$\vec{r}$ denotes the position in space at which the potential is measured and $\vec{r}_i$ denotes the position of the charge. In Cartesian coordinates this becomes

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}} \tag{2}$$

Because we are considering only the x, y plane, $z = 0$ and because the two charges are on the x-axis, then $y_i, z_i = 0$, and $N = 2$

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{\sqrt{(x-x_i)^2+y^2}} \tag{3}$$

1 x-axis

This section looks at the four cases for the potential on the x-axis. since $y = 0$, then for all four cases on the x-axis,

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{\sqrt{(x-x_i)^2}} \tag{4}$$

1.1 2 positive charges, $+Q$, one at D and one at $-D$, $|x| << D$

With both charges equal to $+Q$, Eq. 4 leads to

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-D)^2}}+\frac{1}{\sqrt{(x+D)^2}}\right) \tag{5}$$

Because $|x| << D$, this leads to

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{D-x}+\frac{1}{D+x}\right) \tag{6}$$

Factoring out D from the denominator yields

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{D}\left(\frac{1}{1-\frac{x}{D}}+\frac{1}{1+\frac{x}{D}}\right) \tag{7}$$

Which can be rewritten as

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{D}\left(\left(1-\frac{x}{D}\right)^{-1}+\left(1+\frac{x}{D}\right)^{-1}\right) \tag{8}$$

Using the power series $(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + ...$ results in

$$\begin{aligned} V(x,y,z) = & \quad \frac{Q}{4\pi\epsilon_0}\frac{1}{D}\left(1 + \frac{x}{D} + \frac{x^2}{D^2} + \frac{x^3}{D^3} + ...\right) \\ & + \frac{Q}{4\pi\epsilon_0}\frac{1}{D}\left(1 - \frac{x}{D} + \frac{x^2}{D^2} - \frac{x^3}{D^3} + ...\right) \end{aligned} \tag{9}$$

The odd powers cancel to produce the expansion

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{2}{D}\left(1 + \frac{x^2}{D^2} + \frac{x^4}{D^4} + ...\right) \tag{10}$$

1.2 Opposite charges, $+Q$ at $+D$, $-Q$ at $-D$, $|x| << D$

Eq. 4 now leads to the same results for Eq. 5 except for a sign change, becoming

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-D)^2}} - \frac{1}{\sqrt{(x+D)^2}}\right) \tag{11}$$

Using the same procedure as in Eq.6 - 9 before, we now have

$$\begin{aligned} V(x,y,z) = & \quad \frac{Q}{4\pi\epsilon_0}\frac{1}{D}\left(1 + \frac{x}{D} + \frac{x^2}{D^2} + \frac{x^3}{D^3} + ...\right) \\ & - \frac{Q}{4\pi\epsilon_0}\frac{1}{D}\left(1 - \frac{x}{D} + \frac{x^2}{D^2} - \frac{x^3}{D^3} + ...\right) \end{aligned} \tag{12}$$

Now the even powers cancel to become

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{2}{D}\left(\frac{x}{D} + \frac{x^3}{D^3} + \frac{x^5}{D^5} + ...\right) \tag{13}$$

1.3 2 positive charges, $+Q$, one at D and one at $-D$, $x >> D$

Starting with Eq.5, but now with $x >> D$,

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{x-D} + \frac{1}{x+D}\right) \tag{14}$$

Factoring out x from the denominator yields

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{x}\left(\frac{1}{1-\frac{D}{x}} + \frac{1}{1+\frac{D}{x}}\right) \tag{15}$$

Using the Laurent series expansion now results in

$$\begin{aligned} V(x,y,z) = & \quad \frac{Q}{4\pi\epsilon_0}\frac{1}{x}\left(1 + \frac{D}{x} + \frac{D^2}{x^2} + \frac{D^3}{x^3} + ...\right) \\ & + \frac{Q}{4\pi\epsilon_0}\frac{1}{x}\left(1 - \frac{D}{x} + \frac{D^2}{x^2} - \frac{D^3}{x^3} + ...\right) \end{aligned} \tag{16}$$

The odd powers of the expansion cancel to become

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{2}{x}\left(1 + \frac{D^2}{x^2} + \frac{D^4}{x^4} + ...\right) \tag{17}$$

However, it should noted that this is an odd function, and multiplying through by $\frac{1}{x}$ results in

$$V(x,y,z) = \frac{2Q}{4\pi\epsilon_0}\left(\frac{1}{x} + \frac{D^2}{x^3} + \frac{D^4}{x^5} + ...\right) \tag{18}$$

1.4 Opposite charges, $+Q$ at $+D$, $-Q$ at $-D$, $x >> D$

Changing the sign in Eq. 14 results in the even powers of the expansion cancelling and

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{2}{x}\left(\frac{D}{x} + \frac{D^3}{x^3} + \frac{D^5}{x^5} + ...\right) \tag{19}$$

Which can be rewritten as

$$V(x,y,z) = \frac{2Q}{4\pi\epsilon_0}\left(\frac{D}{x^2} + \frac{D^3}{x^4} + \frac{D^5}{x^6} + ...\right) \tag{20}$$

2 y-axis

This section looks at the four cases for the potential on the y-axis, where we now consider that $x = 0$ and Eq. 3 becomes

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{\sqrt{x_i^2 + y^2}} \tag{21}$$

Because $x_i = \pm D$

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{\sqrt{D^2 + y^2}} \tag{22}$$

2.1 2 positive charges, $+Q$, one at D and one at $-D$, $|y| << D$

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{2Q}{\sqrt{D^2 + y^2}} \tag{23}$$

Factor out D from the denominator yields

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{1}{D}\frac{2Q}{\sqrt{1 + \frac{y^2}{D^2}}} \tag{24}$$

Which can be rewritten as

$$V(x,y,z) = \frac{2Q}{4\pi\epsilon_0}\frac{1}{D}\left(1 + \frac{y^2}{D^2}\right)^{\frac{-1}{2}} \tag{25}$$

Using the power series $(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + ...$ results in

$$V(x,y,z) = \frac{2Q}{4\pi\epsilon_0}\frac{1}{D}\left(1 - \frac{1}{2}\frac{y^2}{D^2} + \frac{3}{8}\frac{y^4}{D^4} + ...\right) \tag{26}$$

2.2 Opposite charges, $+Q$ at $+D$, $-Q$ at $-D$, either $|x| << D$ or $x >> D$

Either inspection or calculation reveals that the potential is always zero on the y-axis for this case

$$V(x, y, z) = 0 \tag{27}$$

2.3 2 positive charges, $+Q$, one at D and one at $-D$, $y >> D$

Beginning with Eq. 24, Factoring out y from the denominator yields

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\frac{1}{y}\frac{2Q}{\sqrt{1 + \frac{D^2}{y^2}}} \tag{28}$$

Following the same method as in Eq. 25 and 26 results is the Laurent series expansion

$$V(x, y, z) = \frac{2Q}{4\pi\epsilon_0}\frac{1}{y}\left(1 - \frac{1}{2}\frac{D^2}{y^2} + \frac{3}{8}\frac{D^4}{y^4} + ...\right) \tag{29}$$