PH 314 Homework Set #5 Solutions

1) $m_\pi = 135\ MeV/c^2$ proton $\rightarrow$ proton + pion

(a) $\Delta E = \Delta m c^2 = m_\pi c^2 = (135\ MeV/c^2)\,c^2 = 135\ MeV$

(b) $\Delta E \Delta t \sim \hbar$ $\quad \Delta t \sim \dfrac{\hbar}{\Delta E} = \dfrac{6.58 \times 10^{-16}\ eV \cdot s}{135 \times 10^{6}\ eV} = 4.87 \times 10^{-24}\ s$

(c) $v = \dfrac{\Delta x}{\Delta t}$ $\quad \Delta x = v \Delta t = c \Delta t = (2.998 \times 10^{8})(4.874 \times 10^{-24})$

$\Delta x = 1.46 \times 10^{-15}\ m = 1.46\ fm$

2) $\omega = A(\sin Bk)$ $\quad A > 0,\ B > 0,\ 0 < Bk < \pi/2$ rad.

(a) Units of ω (angular frequency) = rad/s
Units of $(\sin Bk)$ = none.
Therefore, units of A = rad/s

Units of Bk = rad
Units of k (wave number) = rad/m
Therefore, units of B = meters.

(b) $v(\text{group}) = \dfrac{d\omega}{dk} = \dfrac{d}{dk}\left(A(\sin Bk)\right) = AB\cos(Bk)$

$v(\text{phase}) = \dfrac{\omega}{k} = \dfrac{A(\sin Bk)}{k}$

(c) $k = \dfrac{2\pi}{\lambda}$ $\quad$ If $\lambda \rightarrow \infty$, then $k \rightarrow 0$.

$\lim_{k \to 0} v(\text{group}) = \lim_{k \to 0} AB\cos(Bk) = AB.$

$\lim_{k \to 0} v(\text{phase}) = \lim_{k \to 0} \dfrac{A(\sin Bk)}{k} = A \lim_{k \to 0} \left(\dfrac{\sin Bk}{k}\right)$

This is an indeterminate form. Use L'Hospital's Rule.

$$\lim_{k \to 0} \left(\frac{\sin Bk}{k} \right) = \lim_{k \to 0} \left(\frac{B \cos Bk}{1} \right) = B$$

$$\lim_{k \to 0} v(\text{phase}) = A \lim_{k \to 0} \left(\frac{\sin Bk}{k} \right) = AB.$$

In the long-wavelength limit, the two speeds are the same.

3) $y(x) = -A\cos\left(\frac{2\pi x}{\lambda} + \frac{\pi}{6} \right) = -A\cos\left(kx + \frac{\pi}{6}\right) \quad (x<0)$

$y'(x) = +A'\cos\left(\frac{4\pi x}{\lambda} + \phi' \right) = +A'\cos(2kx + \phi') \quad (x>0)$

At $x=0$, $y(x) = y'(x)$ and $\frac{dy}{dx} = \frac{dy'}{dx}$

$$y(x=0) = -A\cos\frac{\pi}{6} = -\frac{\sqrt{3}A}{2}$$

$$y'(x=0) = +A'\cos\phi'$$

$$\frac{dy}{dx}(x=0) = A\left(\frac{2\pi}{\lambda} \sin\frac{\pi}{6} \right) = A\left(\frac{2\pi}{\lambda}\frac{1}{2} \right) = \frac{\pi A}{\lambda}$$

$$\frac{dy'}{dx}(x=0) = -A'\left(\frac{4\pi}{\lambda}\sin\phi' \right) = \frac{-4\pi A'}{\lambda}\sin\phi'$$

$$-\frac{\sqrt{3}A}{2} = A'\cos\phi' \qquad \text{and} \qquad \frac{\pi A}{\lambda} = \frac{-4\pi A'}{\lambda}\sin\phi'$$

$$\cos\phi' = \frac{-\sqrt{3}A}{2A'} \qquad \text{and} \qquad \sin\phi' = \frac{-A}{4A'}$$

$$(\sin\phi')^2 + (\cos\phi')^2 = 1$$

$$\left(\frac{-A}{4A'} \right)^2 + \left(\frac{-\sqrt{3}A}{2A'} \right)^2 = 1 \qquad \frac{A^2}{16A'^2} + \frac{3A^2}{4A'^2} = 1$$

$$A^2 + 12A^2 = 16A'^2 \qquad 13A^2 = 16A'^2 \qquad A' = \frac{\sqrt{13}}{4}A, -\frac{\sqrt{13}}{4}A$$

Restrict ϕ' to $0 < \phi' < \pi/2$ rad. Then, $A' = -\frac{\sqrt{13}}{4}A = -0.901A$

$$\sin\phi' = \frac{-A}{4A'} \qquad \phi' = \sin^{-1}\left(\frac{-A}{4A'}\right) = \sin^{-1}\left(\frac{-A}{\left(-\frac{\sqrt{13}}{4}A\right)(4)}\right)$$

$$\phi' = \sin^{-1}\left(\frac{1}{\sqrt{13}}\right) = 16.1^\circ$$

4) $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \quad (n = 1, 2, 3, \ldots)$ Ground State $n = 1$

$$\psi_1(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$$

(a) $P(0 < x < \frac{L}{3}) = \int_0^{L/3} |\psi_1(x)|^2 dx = \int_0^{L/3} \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right) dx$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad \sin^2\left(\frac{\pi x}{L}\right) = \frac{1 - \cos\left(\frac{2\pi x}{L}\right)}{2}$$

$$P(0<x<\tfrac{L}{3}) = \frac{2}{L}\int_0^{L/3}\left(\frac{1 - \cos(2\pi x/L)}{2}\right)dx = \frac{2}{L}\left[\frac{x}{2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_0^{L/3}$$

$$P(0<x<\tfrac{L}{3}) = \left[\frac{x}{L} - \frac{1}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_0^{L/3} = \frac{1}{3} - \frac{1}{2\pi}\sin\left(\frac{2\pi}{3}\right)$$

$$P(0<x<\tfrac{L}{3}) = \frac{1}{3} - \frac{1}{2\pi}\frac{\sqrt{3}}{2} = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0.1955$$

(b) $P(\frac{L}{3} < x < \frac{2L}{3}) = \left[\frac{x}{L} - \frac{1}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_{L/3}^{2L/3} = \frac{2}{3} - \frac{1}{2\pi}\sin\left(\frac{4\pi}{3}\right) - \left(\frac{1}{3} - \frac{1}{2\pi}\sin\left(\frac{2\pi}{3}\right)\right)$

$$P(\tfrac{L}{3} < x < \tfrac{2L}{3}) = \frac{1}{3} + \frac{\sqrt{3}}{4\pi} + \frac{\sqrt{3}}{4\pi} = 0.6090$$

(c) $P(0<x<\frac{L}{3}) + P(\frac{L}{3}<x<\frac{2L}{3}) + P(\frac{2L}{3}<x<L) = 1$

$P(\frac{2L}{3}<x<L) = 1 - 0.1955 - 0.6090 = 0.1955$ (or by symmetry).

5) $\psi(x) = A \sin x \cos x = \frac{A}{2}(2\sin x \cos x) = \frac{A}{2}\sin 2x \qquad (A>0)$

(a) $K\psi = \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} \qquad \frac{d\psi}{dx} = +A\cos 2x \quad \frac{d^2\psi}{dx^2} = -2A\sin 2x$

$K\frac{A}{2}\sin 2x = \frac{-\hbar^2}{2m}(-2A\sin 2x) \qquad K = \frac{2\hbar^2}{m}$

(b) $K = \frac{p^2}{2m} = \frac{2\hbar^2}{m} \qquad p^2 = 4\hbar^2 \qquad p = 2\hbar$

(c) $\int_{-L}^{L}|\psi(x)|^2 dx = 1 \qquad \frac{A^2}{4}\int_{-L}^{L}(\sin 2x)^2 dx = 1$

$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin^2(2x) = \frac{1-\cos 4x}{2}$

$\int_{-L}^{L}\left(\frac{1-\cos 4x}{2}\right)dx = \frac{4}{A^2} \qquad \left[\frac{x}{2} - \frac{\sin 4x}{8}\right]_{-L}^{L} = \frac{4}{A^2}$

$\left(\frac{L}{2} - \frac{\sin 4L}{8}\right) - \left(-\frac{L}{2} - \frac{\sin(-4L)}{8}\right) = L - \frac{\sin 4L}{8} + \frac{\sin(-4L)}{8}$

$\sin(-4L) = -\sin 4L$ since sine is an odd function.

$L - \frac{\sin 4L}{8} - \frac{\sin 4L}{8} = L - \frac{\sin 4L}{4} = \frac{4L - \sin 4L}{4} = \frac{4}{A^2}$

$A^2 = \frac{16}{4L - \sin 4L}$ Boundary Conditions: $\psi(x=-L) = \psi(x=L) = 0$

$\psi(x=-L) = \frac{A}{2}\sin(-2L) = \psi(x=L) = \frac{A}{2}\sin(2L) = 0$

$$\frac{A}{2}\sin 2L = 0 \qquad \sin 2L = 0$$

$$\sin 4L = \sin(2(2L)) = 2\sin 2L\cos 2L = 0 \qquad \sin 4L = 0$$

$$A^2 = \frac{16}{4L - \sin 4L} = \frac{16}{4L - 0} = \frac{4}{L} \qquad A = \frac{2}{\sqrt{L}}$$