previously synchronized their timers). Their two views of Jill's timer indicate that her timer advanced only 15 nanoseconds. Jack and Fred describe Jill's timer as running slow. The situation is different for Jill, because she looks at two *different* timers, Jack's and Fred's. The asymmetry lies in the fact that there are two observers sitting in the room (Jack and Fred), but only one observer moving through the room (Jill).

On the other hand, suppose that Jill sits at the front of a space ship moving past Jack, and her friend Sally sits further back in the space ship. Jill notes the time on her own timer and on Jack's timer when she passes Jack, and Sally does the same when she passes Jack. When Jill and Sally compare notes, they will describe Jack's timer as seeming to run slow.

21.7 MAGNETIC TORQUE

A magnetic field can cause a current-carrying coil of wire to twist. You may have seen this for yourself, if you were able to suspend a coil of wire and see it twist in the magnetic field of a bar magnet (or you may have floated a coil and seen it twist in the magnetic field of the Earth). Moreover, we have seen that a compass needle can be thought of as a collection of atomic current loops, and you have certainly seen a compass needle twist in a magnetic field.

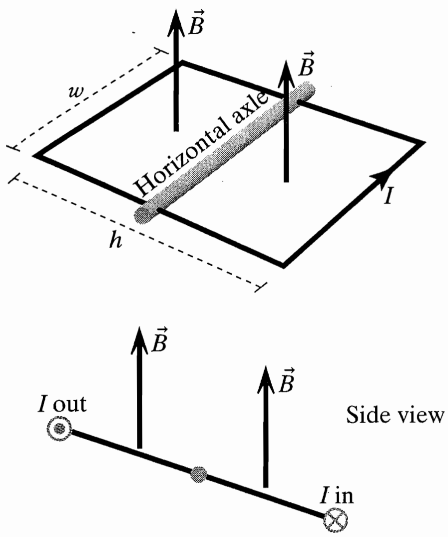

Figure 21.39 A current-carrying loop connected to an insulating horizontal axle is free to rotate.

QUESTION In Figure 21.39, what is the direction of the magnetic force on each of the four sides of the loop?

Consider a rectangular current-carrying loop of wire in a uniform magnetic field (that is, the magnetic field has the same direction and magnitude throughout this region). The loop is free to rotate on an insulating horizontal axle (Figure 21.39).

On the sides of length h the magnetic force is horizontal and points outward, tending to stretch the loop. On the other sides, of length w, the magnetic forces are horizontal and tend to make the loop twist on the axle. This may be easiest to see in the side view shown in Figure 21.39.

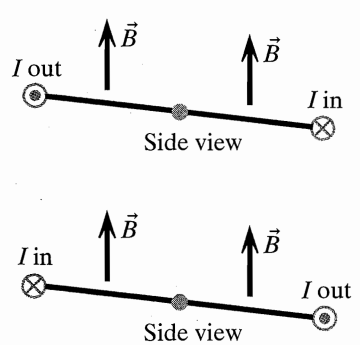

Figure 21.40 Which of these orientations is stable?

QUESTION In the side view, will the loop rotate clockwise ("to the right") or counterclockwise ("to the left")?

The magnetic forces act to twist the loop counterclockwise.

QUESTION When the plane of the loop is perpendicular to the magnetic field, the magnetic forces don't exert any twist. However, one of these orientations is unstable: if you nudge it slightly, the loop will flip over. Which of the two orientations shown in side view in Figure 21.40 is stable, and which is unstable?

The situation on the top is stable: the magnetic forces twist the loop back toward the horizontal plane. The situation on the bottom is unstable: a small displacement away from the horizontal plane leads to magnetic forces that rotate it even farther out of the plane.

It is often easier to talk about the twist on a current-carrying loop of wire in terms of the "magnetic dipole moment" (see Chapter 18). The magnetic dipole moment $\vec{\mu}$ of a current-carrying loop of wire is defined as a vector pointing in the direction of the magnetic field that the loop makes along its axis, which is given by a right-hand rule (this magnetic field made by the loop is of course different from the applied magnetic field that makes the loop twist). The magnitude of the magnetic dipole moment is $\mu = IA$.

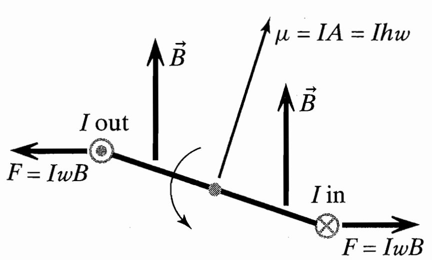

Figure 21.41 The magnetic dipole moment of the loop is a vector that tends to line up with the applied magnetic field.

Our rectangular loop has an area $A = wh$, so its magnetic dipole moment is $\mu = IA = Ihw$. In Figure 21.41, note that the coil tends to twist in a direction to make $\vec{\mu}$ line up with the applied magnetic field $\vec{B}$.

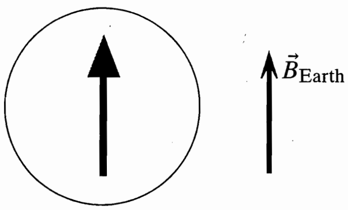

Figure 21.42 What is the direction of the atomic magnetic dipole moments inside this compass needle, which points north?

21.X.13 Inside the magnetized compass needle shown in Figure 21.42, which points north, what is the direction of a typical atomic magnetic dipole moment $\vec{\mu}$?

21.X.14 If the needle is moved away from north and then released, what will happen? Explain briefly.

At last we have an explanation in terms of the fundamental principles of magnetism for the behavior of a compass needle, and why it always lines up with the applied magnetic field. (A practical compass has some friction to damp out the oscillations: without friction the needle would swing back and forth forever, like a frictionless pendulum. Many compasses are filled with liquid to damp out the oscillations.)

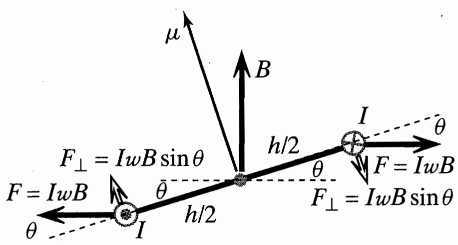

Figure 21.43 The magnetic forces on this rectangular current-carrying loop tend to twist it clockwise (there is a torque $\vec{\mu} \times \vec{B}$ into the page).

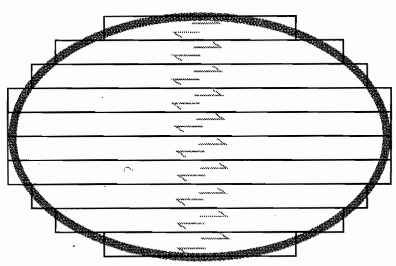

Figure 21.44 A loop of any shape can be approximated by a set of rectangular loops.

Quantitative Torque

The fact that a magnetic dipole twists in a magnetic field is very important. The torque provided by each of the magnetic forces around the axle is equal to the distance from the axle (the "lever arm") times the component of the force perpendicular to the lever arm. The twist applied to this rectangular loop is due to the magnetic forces on the w-lengths of the loop, as is seen in Figure 21.43, where the magnetic torque is out of the page (there is a clockwise twist).

The perpendicular component of each force is $F_\perp = IwB\sin\theta$, and the lever arm is $h/2$. (Note that the other component of F, and the forces on the h lengths of the loop, just tend to stretch the loop without causing a twist.) Each of the forces exerts a torque of $(F_\perp)(h/2)$, so the total torque is $2IwB\sin\theta(h/2) = (Iwh)B\sin\theta$.

Since $\mu = Iwh$, we can express the torque in the form $\vec{\mu} \times \vec{B}$, because this cross product has a magnitude $|\vec{\mu}||\vec{B}|\sin\theta = \mu B\sin\theta$. The direction of this torque vector is along the axle around which the loop rotates. If the thumb of your right hand points in the direction of $\vec{\mu} \times \vec{B}$, your fingers curl around in the direction that the coil will twist. In Figure 21.43, the torque vector is into the page, and the coil rotates clockwise.

We showed that the torque is $\mu B\sin\theta$ for a rectangular loop, but a loop of any shape can be approximated by a set of current-carrying rectangles along whose adjacent sides the current cancels (Figure 21.44). This shows that the torque is $\mu B\sin\theta$ for any kind of loop.

TORQUE ON A MAGNETIC DIPOLE

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

21.8 POTENTIAL ENERGY FOR A MAGNETIC DIPOLE

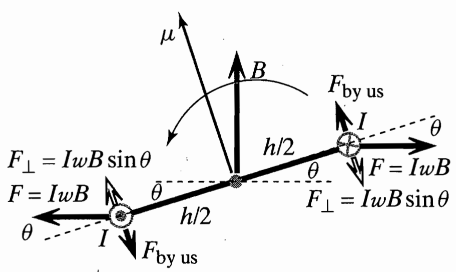

Figure 21.45 Calculating how much work is required to turn a current loop in a magnetic field.

If allowed to pivot freely, a magnetic dipole turns toward alignment with the applied magnetic field. That means that an aligned magnetic dipole is associated with lower potential energy in the magnetic field, and the system tries to go to the lower potential-energy configuration. If we calculate how much work it takes to move the magnetic dipole out of alignment, we will have a measure of the increased potential energy associated with being out of alignment.

We consider the rectangular loop shown in Figure 21.45, and we calculate how much work we have to do to move it from an initial angle $\theta = \theta_i$ to a final angle $\theta = \theta_f$. We move slowly, without changing the kinetic energy of the loop. The loop is always very nearly in equilibrium.

As we rotate the loop through a small angle $d\theta$, both of the forces that we exert act through a distance $(h/2)d\theta$, because arc length is radius $(h/2)$ times angle ($d\theta$ measured in radians). If we force the loop to rotate from the initial angle $\theta = \theta_i$ to a final angle $\theta = \theta_f$, the amount of work we do by exerting our two forces must be calculated by an integral, because F isn't constant. The work that we do goes into changing the magnetic potential energy U_m of the system consisting of the magnetic dipole in the magnetic field:

$$\text{work} = \Delta U_m = \int_{\theta_i}^{\theta_f} 2IwB\sin\theta\left(\frac{h}{2}d\theta\right) = IwhB\int_{\theta_i}^{\theta_f}\sin\theta\, d\theta$$

$$\Delta U_m = IwhB\big[-\cos\theta\big]_{\theta_i}^{\theta_f} = -IwhB[\cos\theta_f - \cos\theta_i]$$

$$\Delta U_m = \Delta(-\mu B\cos\theta) \text{ since } \mu = IA = Iwh$$

It is customary to define the potential energy of a magnetic dipole in a magnetic field by choosing the zero of potential energy in such a way that one writes this:

$$U_m = -\mu B\cos\theta$$

This can also be written as a dot product:

THE POTENTIAL ENERGY FOR A MAGNETIC DIPOLE

$$U_m = -\vec{\mu} \bullet \vec{B}$$

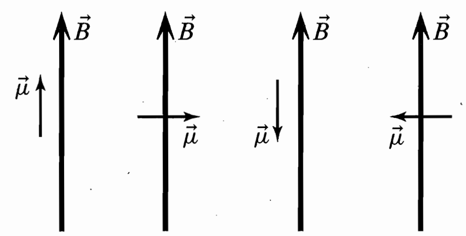

Figure 21.46 Which alignment represents the lowest potential energy? The highest?

This convention corresponds to calling the potential energy zero when θ = 90°, since cos(90°) = 0.

QUESTION Given that $U_m = -\vec{\mu} \bullet \vec{B}$, which of the alignments in Figure 21.46 corresponds to the lowest possible potential energy? To the highest possible potential energy?

The lowest energy corresponds to $\vec{\mu}$ and $\vec{B}$ pointing in the same direction, so that $U_m = -\mu B\cos 0 = -\mu B$. This is also the most stable configuration; if you twist the magnetic dipole away from this position, it will be twisted back by the magnetic field. The highest energy corresponds to $\vec{\mu}$ and $\vec{B}$ pointing in opposite directions, so that $U_m = -\mu B\cos 180° = +\mu B$. This is an unstable configuration; a slight displacement will lead to a large twist. The other two configurations correspond to medium energy: $U_m = 0$ (because cos 90° = 0).

This picture of the potential energy of a magnetic dipole in a magnetic field is very important in atomic and nuclear physics. In a magnetic field, atoms or nuclei can make transitions between higher and lower energy states that correspond to different orientations of the magnetic dipole moments of the atoms or nuclei in the magnetic field.

For example, in the magnetic imaging equipment now used in hospitals, the patient lies inside a large solenoid. In the presence of the magnetic field of the solenoid, nuclei in the patient's body have slightly different energies depending on whether the nuclear magnetic dipole moment of the nucleus is in the direction of or opposite to the applied field. Sensitive equipment detects transitions between these two slightly different energy levels.

21.X.15 What is the energy difference between the highest and lowest states?

Force on a Magnetic Dipole

We have successfully explained the twist of a magnetic dipole in a magnetic field. There remains the question of how to explain the net force on a magnetic

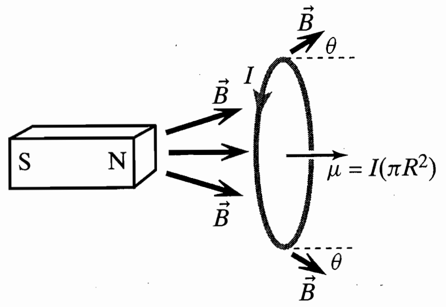

Figure 21.47 A bar magnet exerts a force on a current-carrying loop.

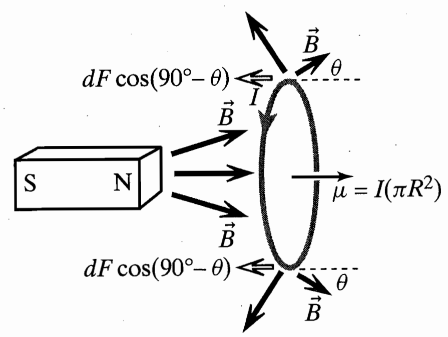

Figure 21.48 Force on a current loop by a bar magnet.

dipole in a magnetic field. For example, how can we explain the attraction or repulsion between two magnets?

We can show that if a magnetic dipole is placed in a nonuniform magnetic field, it experiences a force. Figure 21.47 shows a circular current loop placed near the north pole of a bar magnet, along the axis of the magnet. The magnetic field of the bar magnet diverges with an angle θ as shown and so is nonuniform.

QUESTION Show that the net force on the current loop is to the left, with magnitude $F = I(2\pi R)B\sin\theta = (2\mu B/R)\sin\theta$, where R is the radius of the circular loop.

As shown in Figure 21.48, each segment of the ring experiences a magnetic force $d\vec{F} = I\Delta\vec{l}\times\vec{B}$. The horizontal component of the force is $dF\cos(90°-\theta) = dF\sin\theta$. The vertical components of the forces on two opposite pieces of the ring cancel, so the net force is in the $-x$ direction (toward the bar magnet).

$$F_{\text{net}} = IB\sin\theta\int dl = IB(2\pi R)\sin\theta$$

Since $\mu = IA = I(\pi R^2)$,

$$F_{\text{net}} = \frac{\mu(2B\sin\theta)}{R}$$

If we knew how the magnetic field spread out in angle (that is, how big θ is at a distance R off-axis), we could calculate the net force F. Without that information, we can nevertheless determine F by using an argument based on potential energy, which we will do in the next section.

21.X.16 In what direction would the net force be if the bar magnet were turned around, or the current in the loop reversed?

21.X.17 What can you say about the net force if the field were uniform ($\theta = 0$)?

Calculating the Force on a Magnetic Dipole

Using an energy argument, we can calculate the force on a magnetic dipole due to a nonuniform magnetic field. For example, we can calculate the force on one magnet due to another magnet.

If we move a magnetic dipole from a region of low magnetic field to a region of high magnetic field, the magnetic potential energy $U_m = -\vec{\mu}\bullet\vec{B}$ will change. If released from rest, the object will spontaneously move to the place where the potential energy is lower, picking up kinetic energy as it goes. If we prevent the kinetic energy from changing, by moving the object at a slow constant speed, we have to exert a force and do some work. By calculating how much work we have to do, we can determine how big a force we have to apply, and this is numerically equal to the magnetic force on the object. This will let us calculate how big a force one magnet exerts on another magnet.

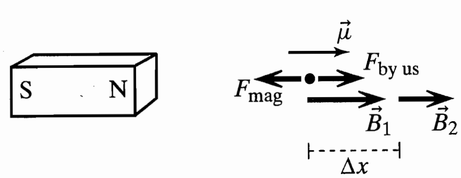

Figure 21.49 A bar magnet acts on a magnetic dipole.

Consider the situation shown in Figure 21.49. A magnetic dipole is aligned with the magnetic field made by a bar magnet, and the magnet exerts a force F_{magnetic} to the left on the magnetic dipole moment, as we saw in the previous section. We exert a force $F_{\text{by us}}$ to the right, infinitesimally larger than F_{magnetic}, and we move the magnetic dipole moment a short distance Δx to the right.

The magnet makes a nonuniform magnetic field (proportional to $1/x^3$ if we are far from the magnet), and we are moving the magnetic dipole from a larger magnetic field B_1 to a smaller magnetic field B_2. The magnetic potential energy increases from $-\mu B_1$ to the somewhat less negative $-\mu B_2$. Therefore we have to do an amount of work

$$F_{\text{by us}}\Delta x = \Delta U_m = (-\mu B_2) - (-\mu B_1) = -\mu(B_2 - B_1) = -\mu\Delta B$$

This is greater than zero because $B_2 < B_1$. The magnitude of the force we have to exert is this:

$$F_{\text{by us}} = \frac{\Delta U_m}{\Delta x} = -\mu\frac{\Delta B}{\Delta x} \rightarrow -\mu\frac{dB}{dx} \quad \text{(a positive value; } dB/dx < 0\text{)}$$

The force that the magnet exerts to the left on the magnetic dipole moment is numerically equal to the force that we exert. This magnitude is also correct in the case of repulsion. If the magnetic dipole moment is at some angle to the magnetic field, there is both a net force and a twist.

This is a special case of the general result that the force associated with a potential energy is the negative gradient of that potential energy:

$$F_x = -\frac{dU}{dx} \quad \text{(general result)}$$

$$F_x = -\frac{d(-\vec{\mu} \bullet \vec{B})}{dx} = \mu\frac{dB}{dx} \quad \text{(our specific case)}$$

In this specific case, the quantity $\vec{\mu} \bullet \vec{B}$ is decreasing in the $+x$ direction, so the equation predicts that the x component of the force on the magnetic dipole moment is in the $-x$ direction, which is correct.

Note that there is no force if the field is uniform ($dB/dx = 0$), in agreement with our previous results, though there is a twist. This is the situation with a compass needle affected by the magnetic field of the Earth, which hardly varies over short distances and therefore applies an extremely weak net force, but there is a twist due to equal and opposite forces that align the needle to point north.

We can now calculate the force that one bar magnet exerts on another. The magnetic field along the axis of one magnet, far from that magnet, is

$$B_1 \approx \frac{\mu_0}{4\pi}\frac{2\mu_1}{x^3}$$

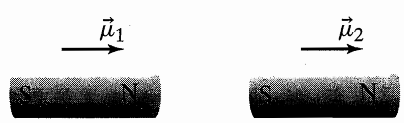

Figure 21.50 The magnetic dipole moment μ_2 of the second magnet lies along the axis of μ_1 of the first magnet.

where μ_1 is the total magnetic dipole moment of all the atomic magnetic dipole moments in the magnet.

QUESTION Show that the attractive force that this magnet would exert on a magnetic dipole moment μ_2 lying along the axis of μ_1 (Figure 21.50) is

$$F \approx 3\mu_2\left(\frac{\mu_0}{4\pi}\frac{2\mu_1}{x^4}\right)$$

Just take the negative gradient of the magnetic potential energy to obtain this result. This says that two magnets should attract or repel each other with a force proportional to $1/x^4$.

Moreover, using our measurements of the magnetic dipole moments of our bar magnets in Chapter 18, we should be able to calculate how close one of our magnets must be to another one in order to pick up the magnet. See Experiment 21.EXP.29.

Magnetic Force and Work

Because the basic magnetic force involves a cross product, the magnetic force $q\vec{v} \times \vec{B}$ on an individual moving charged particle acts perpendicular to the velocity of the particle (and its momentum). Therefore the magnetic force can change the direction of the particle's momentum but not its magnitude or its kinetic energy. Another way of seeing this is to note that the quantity $\vec{F} \cdot d\vec{r} = 0$, so no work is done on the particle.

However, we have also seen more complex situations in which a magnetic force or magnetic torque does do work. Applying a magnetic field can cause a compass needle to begin to rotate; the needle gains rotational kinetic energy. The directions of the magnetic torque and the needle's rotation are the same, so the work done is nonzero. When a bar magnet picks up another bar magnet, the second magnet accelerates upward, gaining kinetic energy. Again, the magnetic force and the displacement are in the same direction, so work is nonzero.

Additionally, if a charged particle is bound to other particles by electric forces, the magnetic force may indirectly result in changes in kinetic energy. Recall that magnetic forces acting on a current of mobile electrons in a wire can indirectly move the stationary positive cores, because the electrons are bound to the positive cores by electric forces, and the battery that drives the current can supply energy.

*The Stern–Gerlach Experiment

An important milestone in our understanding of atoms was the Stern–Gerlach experiment (1922). Silver was vaporized in an oven, and a beam of neutral silver atoms was defined by a series of slits (Figure 21.51). On theoretical grounds, each silver atom was expected to have angular momentum, and a magnetic dipole moment proportional to the angular momentum. The beam of neutral atoms passed through a strongly nonuniform magnetic field, and then struck a cold glass plate onto which the atoms condensed, leaving a visible silvery trace on the glass (The entire apparatus was in a vacuum to avoid collisions with air molecules.)

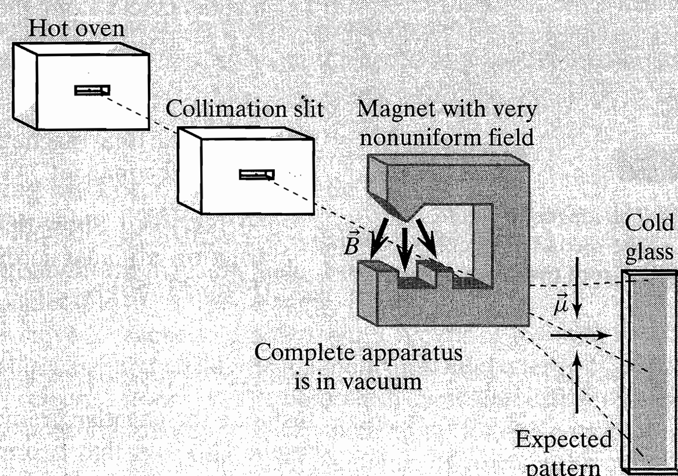

Figure 21.51 Classically, one expects a continuous pattern of silver deposited on the glass.

If the atoms have magnetic dipole moments, they should experience a vertical force, up or down in varying degrees depending on the orientation of their magnetic dipole moments. One would expect a random orientation of the magnetic dipole moments, so that there should be a vertical band of silver deposited on the glass. For example, if the magnetic dipole moment happens to be horizontal, the atom goes straight through without any deflection. (The motion is actually more complicated because the magnetic dipole moment oscillates back and forth in the magnetic field, like the motion around north of an undamped compass needle, unless it happens to point in the direction of the magnetic field. This does not affect the general conclusions, however.)

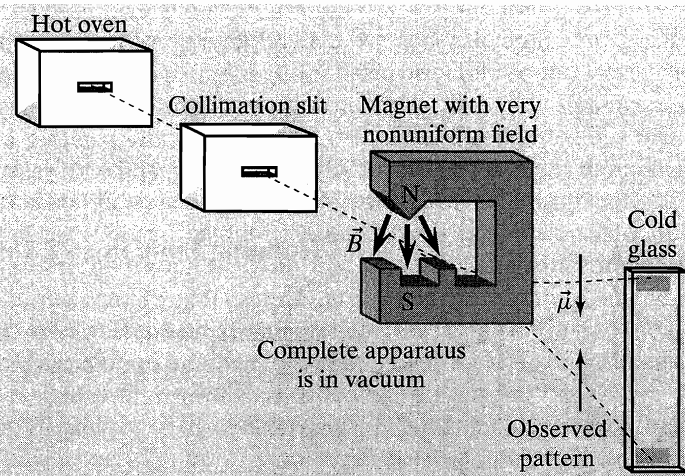

Figure 21.52 What one actually observes is a quantized pattern of silver deposited on the glass.

Instead of being spread out uniformly as one would expect, what Stern and Gerlach actually observed was that the atoms always hit either at the top or at the bottom of the expected region (Figure 21.52). This was interpreted as indicating that the orientation of the angular momentum of the atom is "quantized"—that whenever the atom is observed in this way we find that a component of angular momentum can have only certain quantized values. This distinctive quantization effect is typical of the behavior of atoms and subatomic particles.

21.9 MOTORS AND GENERATORS

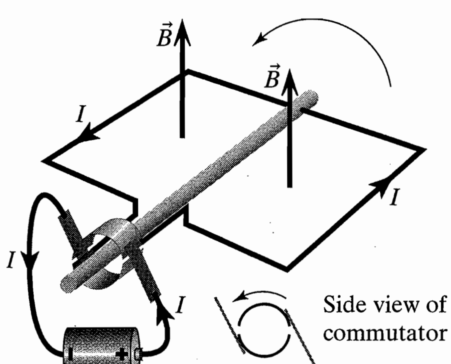

Figure 21.53 A "split-ring commutator" rotates with the coil, and as the coil reaches its most stable position the current in the coil reverses direction.

Electric motors are the most important everyday application of the torque that a magnetic field exerts on a current-carrying coil. Our daily lives are greatly affected by electric motors in refrigerator compressors, water pumps, computer disk drives, VCRs, elevators, automobile starter motors, clocks, and so on.

We have seen that a current-carrying coil twists to its most stable position in a magnetic field. How can we get the coil to turn continuously? The key is to make electrical connections to the coil in such a way that just as it is coming to its stable position, we reverse the direction of the current. Assuming that it is rotating fast enough at that moment, the coil will continue on around in the same direction to the new stable position rather than simply going backward. Then as the coil approaches the new stable position we again switch the direction of the current.

A simple way to achieve continuous rotational motion of the coil is with a "split-ring commutator" that automatically changes the direction of the current through the coil at just the right moment (Figure 21.53). Springy metal tabs or brushes make contact between the battery and the commutator. Whereas actual motors have a wide variety of designs, this simple single-loop motor illustrates the basic principle involved in motors driven by "direct current" (DC, as opposed to alternating current, AC).

Electric Generators

It would be inconvenient to build a generator in the form of a bar riding on rails, such as we considered earlier in this chapter. It is mechanically awkward