Theorems for Analytic Functions

Theorem: $w(z) = u(x,y) + iv(x,y)$ for $z = x + iy$ is analytic in a region $\Leftrightarrow \{u_{,x}, u_{,y}, v_{,x}, v_{,y}\}$ exist, are continuous, and satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Theorem: If $w(z)$ is analytic in a region, then its real u and imaginary v parts each satisfy Laplace's equation in two dimensions, i.e.:

$$\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0$$

Such functions are called *harmonic functions*. Conversely, if $u(x,y)$ is harmonic, then it is the real or imaginary part of a complex analytic function.

Theorem: The Cauchy-Riemann equations hold $\Leftrightarrow \frac{\partial w}{\partial z^*} = 0$.

Theorem: If $w_1(z)$ and $w_2(z)$ are analytic in an open, connected domain D and if $w_1 = w_2$ on a set of points with a limit point z_0 in the interior of D, the $w_1 = w_2$ everywhere in D.