# Series Solutions of Linear ODEs

Consider the standard form of the $2^{nd}$ order, linear, ordinary differential equation:

$$1y'' + P(z)y' + Q(z)y = 0 \qquad (1)$$

(Note the coefficient 1 in front of the second derivative term. Warning: Use this form to classify the singularities, but clear denominators in $P(z)$ and $Q(z)$ before trying a series solution.)

**Theorem:** The solution to equation (1) can only have singularities where the coefficients $P(z)$ and $Q(z)$ have singularities.

If $P(z)$ has a pole of order $m$ and $Q(z)$ has a pole of order $n$, then equation (1) can be rewritten:

$$1y'' + z^{-m}(p_0 + p_1 z + \ldots)y' + z^{-n}(q_0 + q_1 z + \ldots)y = 0 \qquad (2)$$

**Definition:** If $m \leq 1$, $n \leq 2$ at a singularity $z_0$, then $z_0$ is called a *regular singularity* or *regular singular point.*

**Theorem:** If $z_0$ is a *regular singular point*, then then a series solution of equation (1) expanded around $z_0$ consists of either:

1. two Frobenius series,

2. one Frobenius series $y_1(z-z_0)$ and a second solution $y_2(z-z_0) = y_1(z-z)) \ln(z-z_0) + y_0(z-z_0)$ where $y_0(z-z_0)$ is a second Frobenius series.