Chapter 7

Integrating Factors

7.1 WHAT IS AN INTEGRATING FACTOR?

In general, the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{7.1}$$

is not exact. Occasionally, however, it is possible to transform (*7.1*) into an exact differential equation by a judicious multiplication.

***Definition*:** A function $I(x,y)$ is an *integrating factor* for (*7.1*) if the equation

$$I(x,y)[M(x,y)\,dx + N(x,y)\,dy] = 0 \tag{7.2}$$

is exact.

Example 7.1. Determine whether $-1/x^2$ is an integrating factor for $y\,dx - x\,dy = 0$.

Multiplying the given differential equation by $-1/x^2$, we obtain

$$\frac{-1}{x^2}(y\,dx - x\,dy) = 0 \qquad \text{or} \qquad \frac{-y}{x^2}\,dx + \frac{1}{x}\,dy = 0$$

This last equation is exact; hence $-1/x^2$ is an integrating factor.

Example 7.2. Determine whether $-1/xy$ is an integrating factor for $y\,dx - x\,dy = 0$.

Multiplying the given differential equation by $-1/xy$, we obtain

$$\frac{-1}{xy}(y\,dx - x\,dy) = 0 \qquad \text{or} \qquad -\frac{1}{x}\,dx + \frac{1}{y}\,dy = 0$$

This last equation is exact; hence $-1/xy$ is an integrating factor.

Comparing with Example 7.1, we see that a differential equation can have more than one integrating factor.

7.2 SOLUTION BY USE OF AN INTEGRATING FACTOR

If $I(x,y)$ is an integrating factor for (*7.1*), then (*7.2*) is exact and can be solved either by the method of Section 6.2 or, often, by direct integration. The solution of (*7.2*) is also the solution of (*7.1*).

7.3 FINDING AN INTEGRATING FACTOR

It follows from the test for exactness given in Section 6.1 that an integrating factor is a solution to a certain partial differential equation. That equation, however, is usually more difficult to solve than the original differential equation under consideration. Consequently, integrating factors are generally obtained by inspection. The whole success of the method thus depends on the user's ability to recognize or to guess that a particular group of terms composes an exact differential $dh(x, y)$. In this connection, Table 7-1 may prove helpful.

Table 7-1

Group of Terms	Integrating Factor $I(x, y)$	Exact Differential $dh(x, y)$
$y\,dx - x\,dy$	$-\dfrac{1}{x^2}$	$\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
$y\,dx - x\,dy$	$\dfrac{1}{y^2}$	$\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
$y\,dx - x\,dy$	$-\dfrac{1}{xy}$	$\dfrac{x\,dy - y\,dx}{xy} = d\left(\ln \dfrac{y}{x}\right)$
$y\,dx - x\,dy$	$-\dfrac{1}{x^2 + y^2}$	$\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\arctan \dfrac{y}{x}\right)$
$y\,dx + x\,dy$	$\dfrac{1}{xy}$	$\dfrac{y\,dx + x\,dy}{xy} = d(\ln xy)$
$y\,dx + x\,dy$	$\dfrac{1}{(xy)^n}, \quad n > 1$	$\dfrac{y\,dx + x\,dy}{(xy)^n} = d\left[\dfrac{-1}{(n-1)(xy)^{n-1}}\right]$
$y\,dy + x\,dx$	$\dfrac{1}{x^2 + y^2}$	$\dfrac{y\,dy + x\,dx}{x^2 + y^2} = d[\tfrac{1}{2} \ln (x^2 + y^2)]$
$y\,dy + x\,dx$	$\dfrac{1}{(x^2 + y^2)^n}, \quad n > 1$	$\dfrac{y\,dy + x\,dx}{(x^2 + y^2)^n} = d\left[\dfrac{-1}{2(n-1)(x^2 + y^2)^{n-1}}\right]$
$ay\,dx + bx\,dy$ (a, b constants)	$x^{a-1}y^{b-1}$	$x^{a-1}y^{b-1}(ay\,dx + bx\,dy) = d(x^a y^b)$

Sometimes, an integrating factor becomes apparent if the terms of the differential equation are strategically regrouped. (See Problems 7.3–7.5.)

Integrating factors are known if $M(x, y)$ and $N(x, y)$ in (*7.1*) obey certain conditions:

(a) If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) \equiv g(x)$, a function of x alone, then

$$I(x, y) = e^{\int g(x)\,dx} \tag{7.3}$$

(See Problem 7.6.)

(b) If $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) \equiv h(y)$, a function of y alone, then

$$I(x, y) = e^{-\int h(y)\,dy} \tag{7.4}$$

(See Problem 7.8.)

(c) If $M = yf(xy)$ and $N = xg(xy)$, then

$$I(x, y) = \frac{1}{xM - yN} \tag{7.5}$$

(See Problem 7.7.)

Solved Problems

7.1. Solve $y\,dx - x\,dy = 0$.

Using the integrating factor $I(x,y) = -1/x^2$ (see Example 7.1 or Table 7-1), we can rewrite the differential equation as

$$\frac{x\,dy - y\,dx}{x^2} = 0 \tag{1}$$

Since *(1)* is exact, it can be solved by the method of Section 6.2. Alternatively, we note from Table 7-1 that *(1)* can be rewritten as $d(y/x) = 0$. Hence, by direct integration, we have $y/x = c$, or $y = cx$, as the solution.

7.2. Solve the differential equation of Problem 7.1 by using a different integrating factor.

Using the integrating factor $I(x,y) = -1/xy$ (see Example 7.2 or Table 7-1), we can rewrite the differential equation as

$$\frac{x\,dy - y\,dx}{xy} = 0 \tag{1}$$

Since *(1)* is exact, it can be solved by the method of Section 6.2. Alternatively, we note from Table 7-1 that *(1)* can be rewritten as $d[\ln(y/x)] = 0$. Then, by direct integration, $\ln(y/x) = c_1$. Taking the exponential of both sides, we find $\dfrac{y}{x} = e^{c_1}$, or finally,

$$y = cx \qquad (c = e^{c_1})$$

7.3. Solve $(y^2 - y)\,dx + x\,dy = 0$.

No integrating factor is immediately apparent. Note, however, that if terms are strategically regrouped, the differential equation can be rewritten as

$$-(y\,dx - x\,dy) + y^2\,dx = 0 \tag{1}$$

The group of terms in parentheses has many integrating factors (see Table 7-1). Trying each integrating factor separately, we find that the only one that makes the entire equation exact is $I(x,y) = 1/y^2$. Using this integrating factor, we can rewrite *(1)* as

$$-\frac{y\,dx - x\,dy}{y^2} + 1\,dx = 0 \tag{2}$$

Since *(2)* is exact, it can be solved by the method of Section 6.2. Alternatively, we note from Table 7-1 that *(2)* can be rewritten as $-d(x/y) + 1\,dx = 0$, or as $d(x/y) = 1\,dx$. Integrating, we obtain the solution

$$\frac{x}{y} = x + c \qquad \text{or} \qquad y = \frac{x}{x+c}$$

7.4. Solve $(y - xy^2)\,dx + (x + x^2y^2)\,dy = 0$.

No integrating factor is immediately apparent. Note, however, that the differential equation can be rewritten as

$$(y\,dx + x\,dy) + (-xy^2\,dx + x^2y^2\,dy) = 0 \tag{1}$$

The first group of terms has many integrating factors (see Table 7-1). One of these factors, namely $I(x,y) = 1/(xy)^2$, is an integrating factor for the entire equation. Multiplying *(1)* by $1/(xy)^2$, we find

$$\frac{y\,dx + x\,dy}{(xy)^2} + \frac{-xy^2\,dx + x^2y^2\,dy}{(xy)^2} = 0$$

or equivalently,

$$\frac{y\,dx + x\,dy}{(xy)^2} = \frac{1}{x}\,dx - 1\,dy \tag{2}$$