

75. Show that

$$\int \frac{dx}{x\sqrt{x^2-1}} = \begin{cases} \sec^{-1} x + C = \tan^{-1} \sqrt{x^2-1} + C & \text{if } x > 1 \\ -\sec^{-1} x + C = -\tan^{-1} \sqrt{x^2-1} + C & \text{if } x < -1 \end{cases}$$

76. Evaluate for $\int \frac{\sqrt{x^2-1}}{x^3} dx$ for $x > 1$ and for $x < -1$.

77. Graph the function $f(x) = \frac{\sqrt{x^2-9}}{x}$ and consider the region bounded by the curve and the x -axis on $[-6, -3]$. Then, evaluate $\int_{-6}^{-3} \frac{\sqrt{x^2-9}}{x} dx$. Be sure the result is consistent with the graph.

78. Graph the function $f(x) = \frac{1}{x\sqrt{x^2-36}}$ on its domain. Then, find the area of the region R_1 bounded by the curve and the x -axis on $[-12, -12/\sqrt{3}]$ and the region R_2 bounded by the curve and the x -axis on $[12/\sqrt{3}, 12]$. Be sure your results are consistent with the graph.

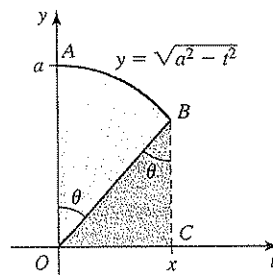
79. **Visual Proof** Let $F(x) = \int_0^x \sqrt{a^2-t^2} dt$. The figure shows that $F(x) = \text{area of sector } OAB + \text{area of triangle } OBC$.

a. Use the figure to prove that $F(x) =$

$$\frac{a^2 \sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2-x^2}}{2}$$

b. Conclude that $\int \sqrt{a^2-x^2} dx =$
 $\frac{a^2 \sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2-x^2}}{2} + C$.

[Source: *The College Mathematics Journal* 34, no. 3 (May 2003)]



QUICK CHECK! ANSWERS

1. Use $x = 3 \sin \theta$ to obtain $9 \cos^2 \theta$. 2. (a) Use $x = 3 \tan \theta$. (b) Use $x = 4 \sin \theta$. 3. Let $x = a \tan \theta$, so that

$dx = a \sec^2 \theta d\theta$. The new integral is $\int \frac{a \sec^2 \theta d\theta}{a^2 (1 + \tan^2 \theta)} =$

$$\frac{1}{a} \int d\theta = \frac{1}{a} \theta + C = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

7.4 Partial Fractions

Recall that a rational function has the form p/q , where p and q are polynomials.

Later in this chapter, we will see that finding the velocity of a skydiver requires evaluating an integral of the form $\int \frac{dv}{a-bv^2}$, where a and b are constants. Similarly, finding the population

of a species that is limited in size involves an integral of the form $\int \frac{dP}{aP(1-bP)}$, where

a and b are constants. These integrals have the common feature that their integrands are rational functions. Similar integrals result from modeling mechanical and electrical networks. The goal of this section is to introduce the *method of partial fractions* for integrating rational functions. When combined with standard and trigonometric substitutions, this method allows us (in principle) to integrate any rational function.

Method of Partial Fractions

Given a function such as

$$f(x) = \frac{1}{x-2} + \frac{2}{x+4},$$

it is a straightforward task to find a common denominator and write the equivalent expression

$$f(x) = \frac{(x+4) + 2(x-2)}{(x-2)(x+4)} = \frac{3x}{(x-2)(x+4)} = \frac{3x}{x^2 + 2x - 8}$$

The purpose of partial fractions is to reverse this process. Given a rational function that is difficult to integrate, the method of partial fractions produces an equivalent function that is much easier to integrate.

<i>rational function</i> $\frac{3x}{(x-2)(x+4)} = \frac{3x}{x^2 + 2x - 8}$	<i>method of partial fractions</i> \longrightarrow	<i>partial fraction decomposition</i> $\frac{1}{x-2} + \frac{2}{x+4}$
Difficult to integrate: $\int \frac{3x}{x^2 + 2x - 8} dx$		Easy to integrate: $\int \left(\frac{1}{x-2} + \frac{2}{x+4} \right) dx$

QUICK CHECK 1 Find an antiderivative of $f(x) = \frac{1}{x-2} + \frac{2}{x+4}$.

The Key Idea Working with the same function, $f(x) = \frac{3x}{(x-2)(x+4)}$, our objective is to write it in the form

$$\frac{A}{x-2} + \frac{B}{x+4}$$

- > Notice that the numerator of the original rational function does not affect the form of the partial fraction decomposition. The constants A and B are called *undetermined coefficients*.

where A and B are constants to be determined. This expression is called the **partial fraction decomposition** of the original function; in this case, it has two terms, one for each factor in the denominator of the original function.

The constants A and B are determined using the condition that the original function f and its partial fraction decomposition must be equal for all values of x ; that is,

$$\frac{3x}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4} \tag{1}$$

- > This step requires that $x \neq 2$ and $x \neq -4$; both values are outside the domain of f .

Multiplying both sides of equation (1) by $(x-2)(x+4)$ gives

$$3x = A(x+4) + B(x-2).$$

Collecting like powers of x results in

$$3x = (A+B)x + (4A-2B). \tag{2}$$

If equation (2) is to hold for all values of x , then

- the coefficients of x^1 on both sides of the equation must match;
- the coefficients of x^0 (that is, the constants) on both sides of the equation must match.

$$3x + 0 = (A+B)x + (4A-2B)$$

This observation leads to two equations for A and B .

Match coefficients of x^1 : $3 = A + B$

Match coefficients of x^0 : $0 = 4A - 2B$

The first equation says that $A = 3 - B$. Substituting $A = 3 - B$ into the second equation gives the equation $0 = 4(3 - B) - 2B$. Solving for B , we find that $6B = 12$, or $B = 2$. The value of A now follows; we have $A = 3 - B = 1$.

Substituting these values of A and B into equation (1), the partial fraction decomposition is

$$\frac{3x}{(x-2)(x+4)} = \frac{1}{x-2} + \frac{2}{x+4}$$

Simple Linear Factors

The previous example illustrates the case of **simple linear factors**; this means the denominator of the original function consists only of linear factors of the form $(x - r)$, which appear to the first power and no higher power. Here is the general procedure for this case.

Like a fraction, a rational function is said to be in **reduced form** if the numerator and denominator have no common factors and it is said to be **proper** if the degree of the numerator is less than the degree of the denominator.

QUICK CHECK 2: If the denominator of a rational function is $(x - 1)(x + 5)(x - 10)$, what is the general form of its partial fraction decomposition?

PROCEDURE Partial Fractions with Simple Linear Factors

Suppose $f(x) = p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q . Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1. Factor the denominator q in the form $(x - r_1)(x - r_2) \cdots (x - r_n)$, where r_1, \dots, r_n are real numbers.

Step 2. Partial fraction decomposition Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}$$

Step 3. Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$, which produces conditions for A_1, \dots, A_n .

Step 4. Solve for coefficients Match like powers of x in Step 3 to solve for the undetermined coefficients A_1, \dots, A_n .

EXAMPLE 1 Integrating with partial fractions

- a. Find the partial fraction decomposition for $f(x) = \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}$.
- b. Evaluate $\int f(x) dx$.

SOLUTION

- a. The partial fraction decomposition is done in four steps.

Step 1: Factoring the denominator, we find that

$$x^3 - x^2 - 2x = x(x + 1)(x - 2)$$

in which only simple linear factors appear.

Step 2: The partial fraction decomposition has one term for each factor in the denominator:

$$\frac{3x^2 + 7x - 2}{x(x + 1)(x - 2)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x - 2} \quad (3)$$

The goal is to find the undetermined coefficients A , B , and C .

Step 3: We multiply both sides of equation (3) by $x(x + 1)(x - 2)$:

$$\begin{aligned} 3x^2 + 7x - 2 &= A(x + 1)(x - 2) + Bx(x - 2) + Cx(x + 1) \\ &= (A + B + C)x^2 + (-A - 2B + C)x - 2A \end{aligned}$$

Step 4: We now match coefficients of x^2 , x^1 , and x^0 on both sides of the equation in Step 3.

$$\text{Match coefficients of } x^2: \quad A + B + C = 3$$

$$\text{Match coefficients of } x^1: \quad -A - 2B + C = 7$$

$$\text{Match coefficients of } x^0: \quad -2A = -2$$

The third equation implies that $A = 1$, which is substituted into the first two equations to give

$$B + C = 2 \quad \text{and} \quad -2B + C = 8$$

You can call the undetermined coefficients A_1, A_2, A_3, \dots or A, B, C, \dots . The latter may be preferable because it avoids subscripts.

Solving for B and C , we conclude that $A = 1$, $B = -2$, and $C = 4$. Substituting the values of A , B , and C into equation (3), the partial fraction decomposition is

$$f(x) = \frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2}.$$

b. Integration is now straightforward:

$$\begin{aligned} \int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} dx &= \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2} \right) dx && \text{Partial fractions} \\ &= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + K && \text{Integrate; arbitrary constant } K. \\ &= \ln \frac{|x|(x-2)^4}{(x+1)^2} + K && \text{Properties of logarithms} \end{aligned}$$

Related Exercises 5–18 ◀

A Shortcut Solving for more than three unknown coefficients in a partial fraction decomposition may be difficult. In the case of simple linear factors, a shortcut saves work. In Example 1, Step 3 led to the equation

$$3x^2 + 7x - 2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1).$$

Because this equation holds for *all* values of x , it must hold for any particular value of x . By choosing values of x judiciously, it is easy to solve for A , B , and C . For example, setting $x = 0$ in this equation results in $-2 = -2A$, or $A = 1$. Setting $x = -1$ results in $-6 = 3B$, or $B = -2$, and setting $x = 2$ results in $24 = 6C$, or $C = 4$. In each case, we choose a value of x that eliminates all but one term on the right side of the equation.

Repeated Linear Factors

The preceding discussion relies on the assumption that the denominator of the rational function can be factored into simple linear factors of the form $(x - r)$. But what about denominators such as $x^2(x - 3)$, or $(x + 2)^2(x - 4)^3$, in which linear factors are raised to integer powers greater than 1? In these cases we have **repeated linear factors**, and a modification to the previous procedure must be made.

Here is the modification: Suppose the factor $(x - r)^m$ appears in the denominator, where $m > 1$ is an integer. Then there must be a partial fraction for each power of $(x - r)$ up to and including the m th power. For example, if $x^2(x - 3)^4$ appears in the denominator, then the partial fraction decomposition includes the terms

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-3)} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3} + \frac{F}{(x-3)^4}.$$

The rest of the partial fraction procedure remains the same, although the amount of work increases as the number of coefficients increases.

PROCEDURE Partial Fractions for Repeated Linear Factors

Suppose the repeated linear factor $(x - r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x - r)$ up to and including the m th power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}$$

where A_1, \dots, A_m are constants to be determined.

► In cases other than simple linear factors, the shortcut can be used to determine some, but not all, of the coefficients, which reduces the work required to find the remaining coefficients.

► *Simple* means the factor is raised to the first power; *repeated* means the factor is raised to a power higher than the first power.

► Think of x^2 as the repeated linear factor $(x - 0)^2$.

EXAMPLE 2 Integrating with repeated linear factors Evaluate $\int f(x) dx$, where

$$f(x) = \frac{5x^2 - 3x + 2}{x^3 - 2x^2}$$

QUICK CHECK 3 State the form of the partial fraction decomposition of the rational function $p(x)/q(x)$ if $q(x) = x^2(x - 3)^2(x - 1)$.

SOLUTION The denominator factors as $x^3 - 2x^2 = x^2(x - 2)$, so it has one simple linear factor $(x - 2)$ and one repeated linear factor x^2 . The partial fraction decomposition has the form

$$\frac{5x^2 - 3x + 2}{x^2(x - 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 2}$$

Multiplying both sides of the partial fraction decomposition by $x^2(x - 2)$, we find

$$\begin{aligned} 5x^2 - 3x + 2 &= Ax(x - 2) + B(x - 2) + Cx^2 \\ &= (A + C)x^2 + (-2A + B)x - 2B \end{aligned}$$

The coefficients A , B , and C are determined by matching the coefficients of x^2 , x^1 , and x^0 :

$$\begin{aligned} \text{Match coefficients of } x^2: & \quad A + C = 5 \\ \text{Match coefficients of } x^1: & \quad -2A + B = -3 \\ \text{Match coefficients of } x^0: & \quad -2B = 2 \end{aligned}$$

Solving these three equations in three unknowns results in the solution $A = 1$, $B = -1$, and $C = 4$. When A , B , and C are substituted, the partial fraction decomposition is

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{4}{x - 2}$$

Integration is now straightforward:

$$\begin{aligned} \int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} dx &= \int \left(\frac{1}{x} - \frac{1}{x^2} + \frac{4}{x - 2} \right) dx && \text{Partial fractions} \\ &= \ln|x| + \frac{1}{x} + 4 \ln|x - 2| + K && \text{Integrate; arbitrary constant } K \\ &= \frac{1}{x} + \ln(|x|(x - 2)^2) + K && \text{Properties of logarithms} \end{aligned}$$

Related Exercises 19–25 ◀

The shortcut can be used to obtain two of the three coefficients easily. Choosing $x = 0$ allows B to be determined. Choosing $x = 2$ determines C . To find A , any other value of x may be substituted.

The quadratic $ax^2 + bx + c$ has no real roots and cannot be factored over the real numbers if $b^2 - 4ac < 0$.

Irreducible Quadratic Factors

It is a fact that a polynomial with real-valued coefficients can be written as the product of linear factors of the form $x - r$ and *irreducible quadratic factors* of the form $ax^2 + bx + c$, where r , a , b , and c are real numbers. By irreducible, we mean that $ax^2 + bx + c$ cannot be factored further over the real numbers. For example, the polynomial

$$x^9 + 4x^8 + 6x^7 + 34x^6 + 64x^5 - 84x^4 - 287x^3 - 500x^2 - 354x - 180$$

factors as

$$\underbrace{(x - 2)}_{\text{linear factor}} \underbrace{(x + 3)^2}_{\text{repeated linear factor}} \underbrace{(x^2 - 2x + 10)}_{\text{irreducible quadratic factor}} \underbrace{(x^2 + x + 1)^2}_{\text{repeated irreducible quadratic factor}}$$

In this factored form, we see linear factors (simple and repeated) and irreducible quadratic factors (simple and repeated).

With irreducible quadratic factors, two cases must be considered: simple and repeated factors. Simple quadratic factors are examined in the following examples, and repeated quadratic factors (which generally involve long computations) are explored in the exercises.

PROCEDURE Partial Fractions with Simple Irreducible Quadratic Factors

Suppose a simple irreducible factor $ax^2 + bx + c$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are unknown coefficients to be determined.

EXAMPLE 3 Setting up partial fractions Give the appropriate form of the partial fraction decomposition for the following functions.

a. $\frac{x^2 + 1}{x^4 - 4x^3 - 32x^2}$

b. $\frac{10}{(x - 2)^2(x^2 + 2x + 2)}$

SOLUTION

- a. The denominator factors as $x^2(x^2 - 4x - 32) = x^2(x - 8)(x + 4)$. Therefore, x is a repeated linear factor, and $(x - 8)$ and $(x + 4)$ are simple linear factors. The required form of the decomposition is

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 8} + \frac{D}{x + 4}$$

We see that the factor $x^2 - 4x - 32$ is quadratic, but it can be further factored, so it is not irreducible.

- b. The denominator is already fully factored. The quadratic factor $x^2 + 2x + 2$ cannot be factored further using real numbers; therefore, it is irreducible. The form of the decomposition is

$$\frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{Cx + D}{x^2 + 2x + 2}$$

Related Exercises 26–29 ◀

EXAMPLE 4 Integrating with partial fractions Evaluate

$$\int \frac{7x^2 - 13x + 13}{(x - 2)(x^2 - 2x + 3)} dx.$$

SOLUTION The appropriate form of the partial fraction decomposition is

$$\frac{7x^2 - 13x + 13}{(x - 2)(x^2 - 2x + 3)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 - 2x + 3}$$

Note that the irreducible quadratic factor requires $Bx + C$ in the numerator of the second fraction. Multiplying both sides of this equation by $(x - 2)(x^2 - 2x + 3)$ leads to

$$\begin{aligned} 7x^2 - 13x + 13 &= A(x^2 - 2x + 3) + (Bx + C)(x - 2) \\ &= (A + B)x^2 + (-2A - 2B + C)x + (3A - 2C) \end{aligned}$$

Matching coefficients of equal powers of x results in the equations

$$A + B = 7 \quad -2A - 2B + C = -13 \quad 3A - 2C = 13.$$

Solving this system of equations gives $A = 5$, $B = 2$, and $C = 1$; therefore, the original integral can be written as

$$\int \frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} dx = \int \frac{5}{x-2} dx + \int \frac{2x+1}{x^2 - 2x + 3} dx.$$

Let's work on the second (more difficult) integral. The substitution $u = x^2 - 2x + 3$ would work if $du = (2x - 2) dx$ appeared in the numerator. For this reason, we write the numerator as $2x + 1 = (2x - 2) + 3$ and split the integral:

$$\int \frac{2x+1}{x^2 - 2x + 3} dx = \int \frac{2x-2}{x^2 - 2x + 3} dx + \int \frac{3}{x^2 - 2x + 3} dx$$

Assembling all the pieces, we have

$$\begin{aligned} \int \frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} dx &= \int \frac{5}{x-2} dx + \int \frac{2x-2}{x^2 - 2x + 3} dx + \int \frac{3}{x^2 - 2x + 3} dx \\ &= 5 \ln |x-2| + \ln |x^2 - 2x + 3| + \frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{x-1}{\sqrt{2}} \right) + C \quad \text{Integrate.} \\ &= \ln |(x-2)^5(x^2 - 2x + 3)| + \frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{x-1}{\sqrt{2}} \right) + C \quad \text{Property of logarithms} \end{aligned}$$

To evaluate the last integral $\int \frac{3 dx}{x^2 - 2x + 3}$, we completed the square in the denominator and used the substitution $u = x - 1$ to produce $\int \frac{3 du}{u^2 + 2}$, which is a standard form.

Related Exercises 30–36 ◀

Final Note The preceding discussion of partial fraction decomposition assumes that $f(x) = p(x)/q(x)$ is a proper rational function. If this is not the case and we are faced with an improper rational function f , we divide the denominator into the numerator and express f in two parts. One part will be a polynomial, and the other will be a proper rational function. For example, given the function

$$f(x) = \frac{2x^3 + 11x^2 + 28x + 33}{x^2 - x + 6}$$

we perform long division:

$$\begin{array}{r} 2x + 13 \\ x^2 - x + 6 \overline{) 2x^3 + 11x^2 + 28x + 33} \\ \underline{2x^3 - 2x^2 + 12x} \\ 13x^2 + 16x + 33 \\ \underline{13x^2 - 13x + 78} \\ 29x - 45 \end{array}$$

It follows that

$$f(x) = \underbrace{2x + 13}_{\substack{\text{polynomial} \\ \text{easy to} \\ \text{integrate}}} + \frac{29x - 45}{x^2 - x + 6} \quad \text{apply partial fraction decomposition}$$

The first piece is easily integrated, and the second piece now qualifies for the methods described in this section.

SUMMARY Partial Fraction Decompositions

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. Simple linear factor A factor $x - r$ in the denominator requires the partial fraction $\frac{A}{x - r}$.

2. Repeated linear factor A factor $(x - r)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_m}{(x - r)^m}$$

3. Simple irreducible quadratic factor An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction

$$\frac{Ax + B}{ax^2 + bx + c}$$

4. Repeated irreducible quadratic factor (See Exercises 67–70.) An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

SECTION 7.4 EXERCISES

Review Questions

- What kinds of functions can be integrated using partial fraction decomposition?
- Give an example of each of the following.
 - A simple linear factor
 - A repeated linear factor
 - A simple irreducible quadratic factor
 - A repeated irreducible quadratic factor
- What term(s) should appear in the partial fraction decomposition of a proper rational function with each of the following?
 - A factor of $x - 3$ in the denominator
 - A factor of $(x - 4)^3$ in the denominator
 - A factor of $x^2 + 2x + 6$ in the denominator
- What is the first step in integrating $\frac{x^2 + 2x - 3}{x + 1}$?

Basic Skills

5–8. Setting up partial fraction decomposition Give the appropriate form of the partial fraction decomposition for the following functions.

5. $\frac{2}{x^2 - 2x - 8}$

6. $\frac{x - 9}{x^2 - 3x - 18}$

7. $\frac{x^2}{x^3 - 16x}$

8. $\frac{x^2 - 3x}{x^3 - 3x^2 - 4x}$

9–18. Simple linear factors Evaluate the following integrals.

9. $\int \frac{dx}{(x - 1)(x + 2)}$

10. $\int \frac{8}{(x - 2)(x + 6)} dx$

11. $\int \frac{3}{x^2 - 1} dx$

12. $\int \frac{dt}{t^2 - 9}$

13. $\int \frac{2}{x^2 - x - 6} dx$

14. $\int \frac{3}{x^3 - x^2 - 12x} dx$

15. $\int \frac{dx}{x^2 - 2x - 24}$

16. $\int \frac{y + 1}{y^3 + 3y^2 - 18y} dy$

17. $\int \frac{1}{x^4 - 10x^2 + 9} dx$

18. $\int \frac{2}{x^2 - 4x - 32} dx$

19–25. Repeated linear factors Evaluate the following integrals.

19. $\int \frac{3}{x^3 - 9x^2} dx$

20. $\int \frac{x}{(x - 6)(x + 2)^2} dx$

21. $\int \frac{x}{(x + 3)^2} dx$

22. $\int \frac{dx}{x^3 - 2x^2 - 4x + 8}$