

Riemann Sheets

Some complex-valued functions $f(z)$ of a complex-variable z are multiple-valued, i.e. for one value z in the domain you get two (or more) different values of $f(z)$ in the range. But then it is impossible to talk about things like continuity or line integrals, because as you move from one value of z to another you don't know which value of $f(z)$ corresponds to a given value of z . What you want is a single-valued function which is as continuous as possible. To do this, you change the **domain**, i.e. imagine a differently shaped surface (no longer the complex plane) which is shaped just right (i.e. like a multi-story parking garage) so that the new function $f(z_{\text{parking garage}})$ can be single valued. The new domain is called a Riemann sheet. ("Height" in the parking garage has no physical significance.)

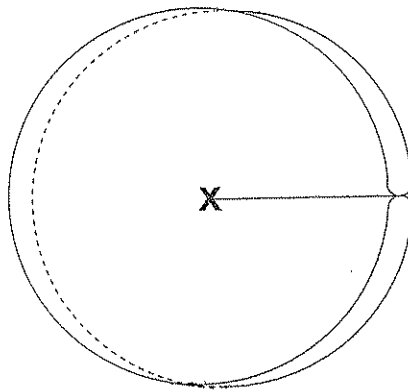
Example: Square Root Riemann Sheet

$$z = r e^{i(\theta + 2n\pi)}$$
$$w = z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i(\frac{\theta}{2} + n\pi)}$$

Branch Point: $z = 0$ has only one square root, so it is clearly a special point, called the branch point.

For $r > 0$ but constant, let θ vary from 0 to 2π , so that z goes once around a circle centered at the branch point. w does **not** come back to the same value. Go twice around, and w **does** come back to the same value.

The Riemann sheet for w looks like:



Branch Cuts and Riemann Surfaces

Finally, let us define the terms *branch point*, *branch line*, and *Riemann surface* by considering the example $w = z^{1/2}$. Let $z = re^{i\theta}$; then there are exactly two different values of w corresponding to any choice of r , θ given by

$$\begin{aligned}w_1 &= r^{1/2}e^{i(\theta/2)} \\w_2 &= r^{1/2}e^{i(\theta/2+\pi)} = -w_1\end{aligned}$$

where, for definiteness, $r^{1/2}$ is taken to be the positive square root of r . The two functions w_1 and w_2 defined in this manner are referred to as *branches* of w . Suppose that we start at some point z_0 and choose one of the two possible values of w —say w_1 . Traverse now a simple closed curve, ending again at z_0 , and requiring w to vary continuously along this curve. It is easily seen that if the curve does not enclose the origin then w returns to the value $w_1(z_0)$, but if the curve does enclose the origin, then w attains the value $w_2(z_0)$. In this latter case, the branches have become interchanged; a second circuit about the origin would change w back to $w_1(z_0)$. The origin is here the only point (apart from the point at ∞) possessing this property that a circuit about it interchanges the branches; such a point is called a *branch point*. In this paragraph, it has been assumed implicitly that the various curves did not pass through the origin, for one could there change arbitrarily from branch to branch without violating the condition of continuous variation of w .

The multiplicity of values of w in such a case as $w = z^{1/2}$ can be awkward, and it may be useful to prevent z from being able to circle the origin. One way of doing this is to "cut" the z plane along the negative real axis and require that no paths in the z plane intersect this barrier. If we then choose the value of w at z_0 to be $w_1(z_0)$ and require that the value of w at any other point z_1 be obtained by the condition of continuous variation of w along any curve joining z_0 and z_1 , then the fact that circuits around the origin are prevented means that w is uniquely defined. Any curve in the z plane originating at the origin and proceeding to infinity would serve equally well as a barrier. Such a barrier curve, constructed for the purpose of ensuring uniqueness, is called a *branch line*, which in this case may be said to be terminated by two points—the origin and the point at infinity.

Let us now allow the point z to move arbitrarily in the z plane, requiring as before that the corresponding value of w vary continuously. Starting with one branch of w , each circuit around the origin will interchange the branches. It may be convenient to think of the z plane as two superposed planes such that, as z circles the origin on one of these planes and starts to repeat its previous curve, the point z now shifts to the other plane (where w has its other branch value); another circuit will again interchange the two planes. For aid in visualization, suppose that the second plane is thought of as a cone with apex angle just less than 180° , with its apex touching the origin and one generator lying along the positive real axis. If we start on the first plane and circle the origin, then z is required to move onto the cone as it crosses the positive real axis; a continued circuit of the origin on the cone surface will again meet the positive real axis, and this time z is required to move back onto the plane. Then for any point z , the corresponding value of w is uniquely defined—one branch value if z is on the lower plane, and the other if z is on the cone. The cone is of course now thought of as being flattened out so as to coincide with the first plane; the use of these two

sheets for the z plane is essentially a topological device for making w single-valued. The resulting two-sheeted range of values for z is called a *Riemann surface* for the function $z^{1/2}$. The reader not familiar with these concepts should pay particular attention to the following exercises.

From "Functions of a Complex Variable"
Carrier, Krook, Pearson pgs 22-24
(McGraw-Hill 1966)