Boundary Conditions for P.D.E.'s

Elliptic Equations

Example: Poisson's Equation

$$\nabla^2\psi(x_k) = f(x_k)$$

Theorem: If $\psi(x_k)$ satisfies Poisson's equation throughout a closed, bounded region R and satisfies Dirichlet conditions on the the boundary ∂R of R, then ψ is unique.

Theorem: If $\psi(x_k)$ satisfies Poisson's equation throughout a closed, bounded region R and satisfies Neumann conditions on the the boundary ∂R of R, then ψ is unique up to an additive constant.

Corollary: If the boundary is piecewise smooth, you can specify either Dirichlet or Neumann conditions on each piece. If Dirichlet conditions are satisfied on at least one piece then ψ is unique.

Corollary: If the region R is unbounded (in some or all directions) but $\psi = o(r^{-1/2})$ as $r \to \infty$ (i.e. ψ falls off faster than $r^{-1/2}$) in the unbounded directions, then ψ is unique.

Parabolic Equations

Example: Inhomogeneous Diffusion Equation

$$\left(\frac{\partial}{\partial t} - k\nabla^2\right)\psi(t, x_k) = f(x_k)$$

Theorem: If $\psi(t, x_k)$ satisfies the inhomogeneous diffusion equation throughout a closed, bounded region R and satisfies either Dirichlet **or** Neumann conditions on the the boundary ∂R of R, **and** ψ satisfies the initial condition

$$\psi(t = 0, x_k) = g(x_k)$$

then ψ is unique.

Corollary: If the spatial boundary is piecewise smooth, you can specify either Dirichlet or Neumann conditions on each piece.

Corollary: If the region R is unbounded (in some or all spatial directions) but $\psi = o(r^{-1/2})$ as $r \to \infty$ (i.e. ψ falls off faster than $r^{-1/2}$) in the unbounded directions, then ψ is unique.

Hyperbolic Equations

Example: Inhomogeneous Wave Equation

$$\left(\frac{-1}{v^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \psi(t, x_k) = f(x_k)$$

Theorem: If $\psi(t, x_k)$ satisfies the inhomogeneous wave equation throughout a closed, bounded region R and satisfies either Dirichlet **or** Neumann conditions on the the boundary ∂R of R, **and** ψ satisfies the two initial conditions

$$\psi(t = 0, x_k) = g(x_k)$$
$$\frac{\partial \psi}{\partial t}(t = 0, x_k) = h(x_k)$$

then ψ is unique.

Corollary: If the spatial boundary is piecewise smooth, you can specify either Dirichlet or Neumann conditions on each piece.

Corollary: If the region R is unbounded (in some or all spatial directions) but $\psi = o(r^{-1/2})$ as $r \to \infty$ (i.e. ψ falls off faster than $r^{-1/2}$) in the unbounded directions, then ψ is unique.