PH461 Math Methods Capstone Homework 2

Due 4/11/16, 4 pm

PRACTICE:

1. **Quiz 2**

   Make sure that you have memorized the following identities and can use them in simple algebra problems:

$$\begin{aligned} e^{u+v} &= e^u\, e^v \\ \ln u\, v &= \ln u + \ln v \\ u^v &= e^{v \ln u} \end{aligned} \tag{1}$$

REQUIRED:

2. **Clean-up from last week**

   Express the complex number $w = \sin^{-1} 3$ in rectangular form.

3. **Derivatives of a Complex Variable**

   (a) Using the definition of derivative, find the region where the function $w(z) = \sin(2z + i)$ is analytic.

   (b) Using the Cauchy-Riemann equations, find the region where the function $w(z) = \sin(2z + i)$ is analytic.

4. **Harmonic Functions**

   Test whether each of the following functions can be the real part of an analytic function. How do you know? In each case, if the given function can be the real part of a an analytic function, find the imaginary part and also find the analytic function itself.

   (a) $f(x, y) = e^{-y} \cos x$

   (b) $f(x, y) = e^{-x} \cos x$

5. **Laplace's Equation in 2-D**

   Prove in an easy way that the function

$$f(x, y) = (x^2 + y^2)^{\frac{1}{4}} \cos(\frac{1}{2} \arctan \frac{y}{x})$$

   is a solution of Laplace's equation (i.e. $\nabla^2 f(x, y) = 0.$) for most values of x and y. For what values is this function not a solution of Laplace's equation?

6. **Poles**
   Find the location and order of any poles for the following functions.

   (a) $\sin z$

   (b) $\frac{z+3}{(z-2i)((z+4)^2}$

   (c) $\frac{\cos z}{z^5}$

   (d) $\frac{z-3}{(z^2-9)^2}$

7. **Conformal Mapping**
   If $f(z)$ is an analytic function, then it maps circles in the complex plane (i.e. the Argand diagram representing the domain of the function) to circles in the complex plane (i.e. the Argand diagram representing the range of the function. An analytic function also preserves angles. Such a map is called a conformal map. Consider the following shapes in the domain:

   Shape 1: $|z| = 2$
   Shape 2: the real axis
   Shape 3: the unit square with lower left corner at the origin

   (a) Transform these shapes using the function $f(z) = z^3$ and draw the transformed shapes in the Argand diagram representing the domain. Is this function analytic?

   (b) Transform these shapes using the function $f(z) = \frac{1}{z}$ and draw the transformed shapes in the Argand diagram representing the domain. Is this function analytic?

8. **Branch Points**
   Consider the functions $w(z) = \sqrt{z^3 + (-2+i)z^2 - 2iz}$.

   (a) Find the branch points of $w(z)$.

   (b) Evaluate $w(-i)$ and $w(1)$ in both rectangular and exponential form. Do these calculations by hand, check using a computer algebra system such as *Mathematica* or *Maple*.

   (c) Show in a sketch at least 2 distinct ways of placing branch lines in order to make $w(z)$ single-valued.