PH461 Math Methods Capstone Homework 8

Due Tuesday 5/23/16, 3:50 pm

PRACTICE:

1. **Quiz 7–See handwritten solutions on last page.**

   Recurrence Relations

   For each of the problems below, suppose you have been solving a differential equation using power series methods around the indicated point. You find the indicated recurrence relation. Write out the first four nonzero terms in the power series expansion. If the recurrence relation allows two solutions, write out the first five nonzero terms in each such solution.

   (a) In an expansion around the point $z = 1$, the recurrence relation is:

   $$a_{n+1} = \frac{1}{n+1}\, a_n$$

   (b) In an expansion around the point $z = 0$, the recurrence relation is:

   $$a_{n+2} = -\frac{(5-n)(6+n)}{(n+2)(n+1)}\, a_n$$

REQUIRED:

2. **Series of Legendre Polynomials**

   Use your favorite tool (*e.g.* Maple, Mathematica, Matlab, pencil) to generate the Legendre polynomial expansion to the function $f(z) = \sin(\pi z)$. How many terms do you need to include in a partial sum to get a "good" approximation to $f(z)$ for $-1 < z < 1$? What do you mean by a "good" approximation? How about the interval $-2 < z < 2$? How good is your approximation? Discuss your answers. Answer the same set of questions for the function $g(z) = \sin(3\pi z)$

3. **Series Solutions**

   Consider the differential equation

   $$y' - 2xy = 0$$

   Find the first three non-zero terms in ONE Frobenius series solution of this differential equation, expanded around $x = 0$. (I don't care which one.)

   (a) Find the general solution of the differential equation in a series solution around the point $x = 1$. You only need to find the first 4 nonzero terms.

   (b) Find the exact, general solution of the differential equation by other methods. Expand your exact solution in a series around the point $x = 1$ and compare your answer to that in part (a).

4. **Series Solutions**
Consider the differential equation

$$8x^2y'' + 10xy' + (x-1)y = 0$$

Find the first three non-zero terms in ONE Frobenius series solution of this differential equation, expanded around $x = 0$. (I don't care which one.)

a) $$a_{n+1} = \frac{1}{n+1} a_n$$

$$a_1 = a_0$$

$$a_2 = \frac{1}{2} a_1 = \frac{1}{2} a_0$$

$$a_3 = \frac{1}{3} a_2 = \frac{1}{3 \cdot 2} a_0$$

$$a_4 = \frac{1}{4} a_3 = \frac{1}{4 \cdot 3 \cdot 2} a_0$$

"Expand around $z=1$" means powers of $(z-1)$

$$y = a_0 \left(1 + (z-1) + \frac{1}{2}(z-1)^2 + \frac{1}{3 \cdot 2}(z-1)^3 + \frac{1}{4 \cdot 3 \cdot 2}(z-1)^4 + \dots\right.$$

b) $$a_{n+2} = -\frac{(5-n)(6+n)}{(n+2)(n+1)} a_n$$

$$a_2 = -\frac{5 \cdot 6}{2 \cdot 1} a_0$$

$$a_4 = -\frac{3 \cdot 8}{4 \cdot 3} a_2 = \frac{5 \cdot 3 \cdot 6 \cdot 8}{2 \cdot 4 \cdot 1 \cdot 3} a_0$$

$$a_6 = -\frac{1 \cdot 10}{6 \cdot 5} a_4 = -\frac{5 \cdot 3 \cdot 1 \cdot 6 \cdot 8 \cdot 10}{2 \cdot 4 \cdot 6 \cdot 1 \cdot 3 \cdot 5} a_0$$

$$a_3 = -\frac{4 \cdot 7}{3 \cdot 2} a_1$$

$$a_5 = -\frac{2 \cdot 9}{5 \cdot 4} a_3 = \frac{4 \cdot 2 \cdot 7 \cdot 9}{3 \cdot 5 \cdot 2 \cdot 4} a_1$$

$0 \cdot 11$ — This series

$$a_7 = -\frac{0 \cdot 11}{7 \cdot 6} a_5 = 0$$ $\leftarrow$ This series terminates

$$y = a_0 \left(1 - \frac{5 \cdot 6}{2 \cdot 1} z^2 + \frac{5 \cdot 3 \cdot 6 \cdot 8}{2 \cdot 4 \cdot 1 \cdot 3} z^4 - \frac{5 \cdot 3 \cdot 1 \cdot 6 \cdot 8 \cdot 10}{2 \cdot 4 \cdot 6 \cdot 1 \cdot 3 \cdot 5} z^6 + \ldots \right)$$

$$+ a_1 \left(z - \frac{4 \cdot 7}{3 \cdot 2} z^3 + \frac{4 \cdot 2 \cdot 7 \cdot 9}{3 \cdot 5 \cdot 2 \cdot 4} z^5 \right)$$