Important Partial Differential Equations in Physics

Laplace's Equation:

Many time-independent problems are described by *Laplace's equation*. This is defined for $\psi = \psi(x, y, z)$ by:

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0. \tag{1}$$

The differential operator, ∇^2, defined by eq.(1) is called the *Laplacian* operator, or just the *Laplacian* for short. Some examples of Laplace's equation are the electrostatic potential in a charge-free region, the gravitational potential in a matter-free region, the steady-state temperature in a region with no heat source, the velocity potential for an incompressible fluid in a region with no vortices and no sources or sinks.

Poisson's Equation:

Poisson's equation is like Laplace's equation except that it allows an inhomogeneous term $f(x, y, z)$ known as the source density. It has the form:

$$\nabla^2\psi = f(x, y, z) \tag{2}$$

.

Schrödinger's Equation:

A great deal of quantum mechanics is devoted to the study of the solutions to the *time-dependent Schrödinger equation*:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x, y, z)\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{4}$$

This equation governs the time dependence of the wave-function of a particle moving in a given potential, $V(x, y, z)$. A special role is played by solutions to (4) that have the simple form: $\psi = \phi(x, y, z)\exp(-iEt/\hbar)$. The function ϕ satisfies the *time-independent Schrödinger equation* of *Schrödinger eigenvalue equation*:

$$-\frac{\hbar^2}{2m}\nabla^2\phi + V(x, y, z)\phi = E\phi. \tag{5}$$

In both of these equations $\hbar$ and m represent real constants. E is a constant that emerges during the separation of variables procedure. i, as usual, satisfies $i^2 = -1$.

The Diffusion Equation:

The partial differential equation governing the concentration of a diffusing substance or the non-steady-state temperature in a region with no heat sources is the *diffusion equation*:

$$\frac{\partial \psi}{\partial t} - \kappa \nabla^2 \psi = 0. \tag{6}$$

κ is a real constant called the *diffusivity*.

The Wave Equation:

Wave propagation, including waves on strings or membranes, pressure waves in gasses, liquids or solids, electromagnetic waves and gravitational waves, and the current or potential along a transmission line all satisfy the following *wave equation*:

$$-\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi = 0. \tag{7}$$

The real constant v can be interpreted as the speed of the corresponding wave.

The Klein-Gordon Equation:

Disturbances travelling through fields that mediate forces with a finite range, satisfy a modification of the wave equation called the *Klein-Gordon equation*. It is given by:

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{8}$$

The coefficients c, $\hbar$ and m all represent constants.

Helmholtz's Equation:

The equation:

$$\nabla^2\psi + k^2\psi = 0 \tag{3}$$

is known as *Helmholtz's equation* and arises as the time-independent part of the diffusion or wave equations. k is a constant that emerges during the separation of variables procedure.