Classification of Partial Differential Equations

The most general 2nd order, linear, PDE can be written:

$$\mathcal{L}\psi = f$$

where

$$\mathcal{L}\psi = \sum_{i,j=1}^{4} A_{ij}(x_k)\frac{\partial^2\psi}{\partial x_i \partial x_j} + \sum_{i=1}^{4} B_i(x_k)\frac{\partial\psi}{\partial x_i} + C(x_k)\psi$$

The PDE is classified according to the signs of the eigenvalues $\lambda_i(x_k)$ of the matrix of functions $A_{ij}(x_k)$.

1. **Elliptic**: $\lambda_i(x_k)$ are nowhere vanishing. All have the same sign.

   Ex: Poisson, Laplace, Helmholtz

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$A_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. **Parabolic**: One eigenvalue vanishes everywhere (usually time dependence), the others are nowhere vanishing and have the same sign.

   Ex: Diffusion, Schroedinger

$$A_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. **Hyperbolic**: All eigenvalues are nowhere vanishing. One sign differs from the others.

   Ex: Wave, Klein-Gordon

$$A_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$