Sturm-Liouville Theory

(boundary value)

A second order Sturm–Liouville problem is a homogeneous boundary value problem of the form

$$\begin{aligned} [P(x)\,y']' + Q(x)\,y + \lambda\,w(x)\,y &= 0 \\ \alpha_1\,y(a) + \beta_1\,y'(a) &= 0 \\ \alpha_2\,y(b) + \beta_2\,y'(b) &= 0 \end{aligned}$$

where P, P', Q, w are continuous and real on $[a, b]$, and P and w are positive.

Theorem: For y_1 and y_2 two linearly independent solutions of the homogeneous differential equation, **nontrivial** solutions of the homogeneous boundary value problem exist iff

$$\det \begin{vmatrix} \alpha_1\,y_1(a) + \beta_1\,y_1'(a) & \alpha_1\,y_2(a) + \beta_1\,y_2'(a) \\ \alpha_2\,y_1(b) + \beta_2\,y_1'(b) & \alpha_2\,y_2(b) + \beta_2\,y_2'(b) \end{vmatrix} = 0$$

Definition: Values of λ for which nontrivial solutions exist are called eigenvalues. The corresponding solutions are called eigenfunctions.

Theorem: The eigenvalues of a homogeneous Sturm-Liouville problem are real and non-negative and can be arranged in a strictly increasing infinite sequence

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \dots$$

and $\lambda_n \to \infty$ as $n \to \infty$.

Theorem: For each eigenvalue, there exists exactly one linearly independent eigenfunction, y_n. These eigenfunctions for differing eigenvalues are orthogonal with respect to the inner product:

$$(y_n, y_m)_w = \int_a^b y_n(x)\, y_m(x)\, w(x)\, dx = N_n \delta_{n,m}$$

Theorem: The eigenfunctions y_n span the vector space of piecewise smooth functions satisfying the boundary conditions of the Sturm-Liouville problem. (Convergence in the mean, not pointwise.)

$$f(x) = \sum_{n=1}^{\infty} c_n\, y_n(x)$$

where the c_n's are given by:

$$c_n = \frac{1}{N_n}(y_n, f)_w = \frac{1}{N_n} \int_a^b y_n^*(x)\, f(x)\, w(x)\, dx$$

Sturm-Liouville Theory

(periodic)

A second order periodic Sturm–Liouville problem is a homogeneous problem of the form

$$[P(x)\,y']' + Q(x)\,y + \lambda\,w(x)\,y = 0$$

where P, P', Q, w are continuous and real on $[a, b]$, and P and w are positive, and

$$\Big[P(x)\left(f^*(x)\,g'(x) - f^{*\prime}(x)\,g(x)\right)\Big]\Big|_a^b = 0$$

for $f(x)$ and $g(x)$ and two vectors in the vector space.

Definition: Values of λ for which nontrivial solutions of the periodic Sturm-Liouville problem exist are called eigenvalues. The corresponding solutions are called eigenfunctions.

Theorem: The eigenvalues of a periodic Sturm-Liouville problem are real.

Theorem: For each eigenvalue, there exist linearly independent eigenfunctions, y_n. These eigenfunctions for differing eigenvalues are orthogonal with respect to the inner product:

$$(y_n, y_m)_w = \int_a^b y_n(x)\, y_m(x)\, w(x)\, dx = N_n \delta_{n,m}$$

Eigenfunctions with the same eigenvalue can be orthogonalized using Gram-Schmidt orthogonalization.

Theorem: The eigenfunctions y_n span the vector space of piecewise smooth functions satisfying the boundary conditions of the Sturm-Liouville problem. (Convergence in the mean, not pointwise.)

$$f(x) = \sum_{n=1}^{\infty} c_n\, y_n(x)$$

where the c_n's are given by:

$$c_n = \frac{1}{N_n}(y_n, f)_w = \frac{1}{N_n}\int_a^b y_n^*(x)\, f(x)\, w(x)\, dx$$