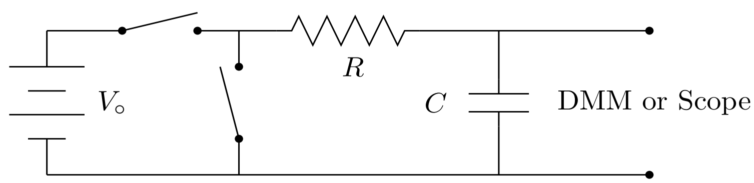
RC Circuits

Concepts

The addition of a simple capacitor to a circuit of resistors allows two related phenomena to occur. The observation that the time-dependence of a complex waveform is altered by the circuit is referred to as a time-domain analysis. On the other hand, observation that a single-frequency wave undergoes an amplitude and phase shift upon passage through the circuit is referred to as a frequency-domain analysis.

Time-Domain Analysis of the RC Circuit



The behavior of this "RC" circuit can be analyzed in the time-domain by solving an appropriate differential equation with the appropriate boundary conditions. Begin with Kirchoff's Potential Law, which is a consequence of conservation of energy

$$V_\circ = V_R + V_C = IR + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C}.$$

Consider the case in which initially the capacitor is charged to $V_\circ$ through the horizontal switch while the vertical switch is open. The boundary or initial condition is that at $t = 0$ $Q = V_\circ/C$. Now the horizontal switch is opened and the vertical switch is closed. Charge flows from one side of the capacitor to the other, and the differential equation to solve is simply

$$V_\circ = V_R + V_C = \frac{dQ}{dt} + \frac{Q}{C} = 0 \rightarrow R\frac{dQ(t)}{dt} = -\frac{Q(t)}{C}.$$

The solution is

$$Q(t) = Q_\circ e^{-\frac{t}{RC}} \text{ with } Q_\circ = V_\circ C.$$

The quantity $\tau = RC$ is the *time constant* or *characteristic time* or *1/e time*, and it is the time for the charge to decay from $Q_\circ$ to $Q_\circ/e$. The potentials across the resistor and capacitor are

$$V_R(t) = I(t)R = R\frac{dQ}{dt} = -V_\circ e^{-\frac{t}{RC}}\ ,$$

and

$$V_C(t) = Q(t)/C = V_\circ e^{-\frac{t}{RC}}\ .$$

Now consider the case in which initially, RC is grounded through closure of the vertical switch. When this switch is opened and the horizontal switched is closed, current begins to flow and the capacitor begins to charge. The initial or boundary condition is that at $t = 0$ $V_R + V_C = 0$, and the equation to solve is

$$V_\circ = V_R + V_C = IR + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C}.$$

The solution is the solution above (the solution to the homogeneous equation) plus whatever is necessary to satisfy the initial condition. Thus, the solution is

$$Q(t) = Q_\circ(1 - e^{-\frac{t}{RC}}) \text{ with } Q_\circ = V_\circ C.$$

In time $\tau = RC$, Q rises from 0 to $Q_\circ(1 - 1/e)$. Notice that the current decreases with time as

$$I(t) = \frac{dQ}{dt} = \frac{V_\circ}{R} e^{-\frac{t}{RC}} \ .$$

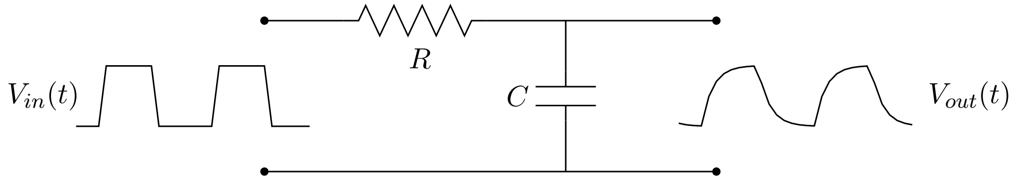
The potentials across the resistor and capacitor are

$$V_R(t) = I(t)R = R\frac{dQ}{dt} = V_\circ e^{-\frac{t}{RC}} \ ,$$

and

$$V_C(t) = Q(t)/C = V_\circ(1 - e^{-\frac{t}{RC}}) \ .$$

Instead of using manual switches, it is easier to use a square wave from a function generator to alternate the applied potential between 0 and $V_\circ$. This circuit and the resulting $V_C(t)$ for a low frequency square wave are shown in the figure below.



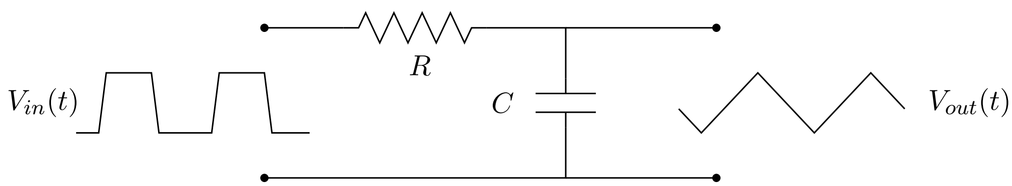
As the frequency of the applied square wave increases, the output waveform will be diminished because there is insufficient time for the capacitor to charge to the applied potential. Examining the expression for $V_C(t)$ at times small compared to τ, we find

$$V_C(t) = V_\circ(1 - e^{-\frac{t}{RC}}) \simeq V_\circ(1 - (1 - \frac{t}{RC})) = V_\circ \frac{t}{RC} \ ,$$

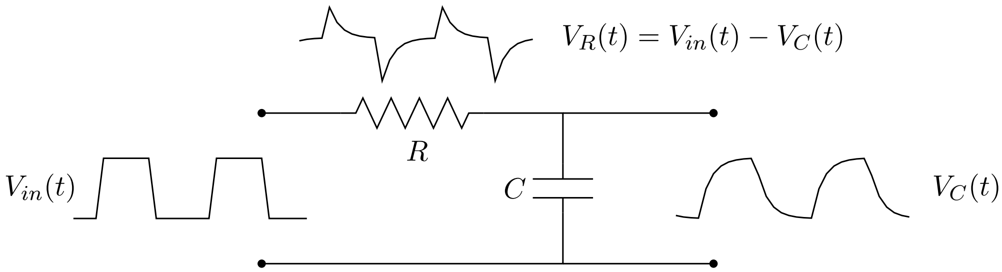
using the expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \simeq 1 + x \ .$$

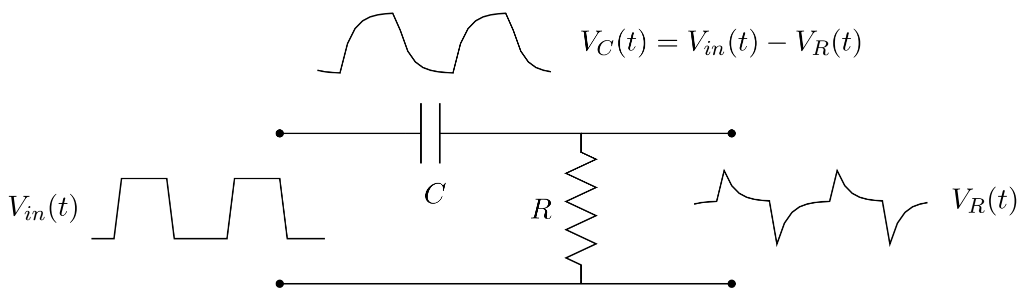
Thus, $V_C(t)$ is linear in t with slope $1/RC$. At high frequencies, the RC circuit acts as an integrator of a square wave or any input waveform.



Since $V_\circ(t) = V_R(t) + V_C(t)$ by conservation of energy, the potential across R is easy to calculate, as shown below. Notice that $V_R(t)$ looks like the derivative of $V_\circ(t)$, and indeed it is for times $t << \tau$.

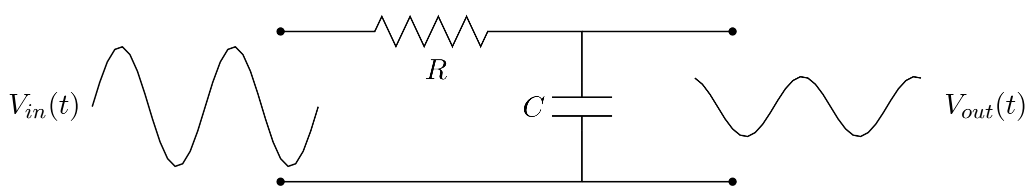


The CR circuit is easy to analyze in the time-domain, again because $V_\circ(t) = V_R(t) + V_C(t)$. The picture below shows the waveforms resulting from the time-domain analysis which we have already performed. The CR circuit acts as a differentiator of an input waveform over times small compared to τ.

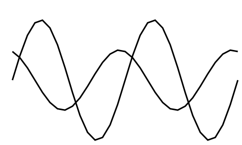


Frequency Domain Analysis

The behavior of the RC circuit in the frequency-domain can be determined by solving the same differential equation using a single-frequency applied potential $V(t) = V_\circ e^{i\omega t}$. As pictured below, the resulting $V_C(t)$ will be of lesser amplitude and shifted in phase.



$A(\omega) = \frac{Max(V_{in}(t))}{Max(V_{out}(t))}$ Notice the phase shift



The analysis begins with

$$V(t) = V_R(t) + V_C(t) = I(t)R + \frac{Q(t)}{C} = R\frac{dQ}{dt} + \frac{Q(t)}{C}.$$

Assuming that $Q(t)$ will also oscillate at the same applied frequency but with a different phase, $Q(t) = Q_\circ e^{i(\omega t+\alpha)}$. So,

$$V_\circ e^{i\omega t} = (i\omega R + \frac{1}{C})Q_\circ e^{i(\omega t+\alpha)}\ ,$$

or

$$V_\circ = (i\omega R + \frac{1}{C})Q_\circ e^{i\alpha} \ .$$

$V_\circ$ is the amplitude of the applied potential at frequency ω, and $Q_\circ/C$ is the amplitude of the signal across the capacitor at frequency ω. At this point, we have two unknowns, $Q_\circ$ and α, but only one equation. However, there are actually two equations here. $V_\circ$ is a real number and must be equal to the real part of the right hand side of the equation, and the imaginary part of the right hand side must be zero. Rewriting this expression as

$$\frac{V_\circ C}{Q_\circ} = (1 + i\omega RC)e^{i\alpha} = \sqrt{1+\omega^2R^2C^2}e^{i\tan^{-1}\omega RC}e^{i\alpha} = \sqrt{1+\omega^2R^2C^2}e^{i(\alpha+\tan^{-1}\omega RC)} \ ,$$

the fact that the imaginary part of the right hand side must be zero yields

$$\alpha = -tan^{-1}\omega RC \ .$$

Then,

$$\frac{V_\circ C}{Q_\circ} = \sqrt{1+\omega^2R^2C^2} \rightarrow Q_\circ = \frac{V_\circ C}{\sqrt{1+\omega^2R^2C^2}} \ .$$

The final result for the signal passing through this circuit is

$$V_C(\omega) = \frac{V_\circ(\omega)}{\sqrt{1+\omega^2R^2C^2}}e^{-i\tan^{-1}\omega RC} \ .$$

The frequency-dependent *transmission function* or *response function* for this circuit is

$$A(\omega) = \frac{V_C(\omega)}{V_\circ(\omega)} = \frac{1}{\sqrt{1+\omega^2R^2C^2}}e^{-i\tan^{-1}\omega RC} \ .$$

Notice that the phase difference $\alpha \rightarrow 0$ as $\omega \rightarrow 0$ and $\alpha \rightarrow -\pi/2$ as $\omega \rightarrow \infty$. and that the amplitude $|A(\omega)|$ is 1 at low frequency but tends to zero at high frequency. The conclusion is that this circuit is a *low pass filter*, meaning that it does not transmit high frequencies.

Concept of Impedance

The current-potential relationship across the capacitor is interesting. The current in the circuit is

$$I(t) = \frac{dQ}{dt} = i\omega Q_\circ e^{i(\omega t+\alpha)} \ ,$$

so, in the frequency domain,

$$I(\omega) = \omega Q_\circ(\omega)e^{i(\alpha+\pi/2)} \ .$$

The interesting result is that

$$\frac{V_C(\omega)}{I(\omega)} = \frac{1}{i\omega C} \ .$$

Thus, the potential across a capacitor is

$$V(\omega) = I(\omega)Z(\omega) \ ,$$

where the *impedance* of the capacitor is

$$Z(\omega) = \frac{1}{i\omega C} \ .$$

The expression $V(\omega) = I(\omega)Z(\omega)$ looks like Ohm's law, but it is not. It simply states that the linear relationship between the frequency-dependent complex potential and the frequency-dependent complex current. Capacitors are non-dissipative elements, so the time-average power $Re(\overline{V(t)I(t)}) = 0$. For a resistor, the impedance is $Z(\omega) = R$, a real quantity independent of frequency.

It is important to understand the physical significance of $Z(\omega)$ for a capacitor. Since $Z \to \infty$ as $\omega \to 0$, a capacitor acts as an open circuit at very low frequencies. That is obvious. What was not obvious previously is that as $\omega \to \infty$, $Z \to 0$, which means that a capacitor becomes a conductive short circuit. These two limiting behaviors are very useful in determining the behavior of complex circuits without writing any equations.

Using Impedance in RC and CR Circuit Analyses

With this definition of the complex impedance of a capacitor, it is now easy to analyse the RC circuit in the frequency domain. The expression for a *potential divider* can be used to determine the potential across the capacitor in the simple RC circuit above:

$$V_C(\omega) = V_\circ(\omega)\frac{Z_C(\omega)}{Z_C(\omega) + R} = V_\circ(\omega)\frac{\frac{1}{i\omega C}}{\frac{1}{i\omega C} + R} = V_\circ(\omega)\frac{1}{1 + i\omega RC} = V_\circ(\omega)\frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} e^{-i\tan^{-1}\omega RC} \; .$$

This expression is consistent with our previous conclusion that the RC circuit is a *low pass filter*.

For the CR circuit, the *potential divider expression* yields the potential across the resistor:

$$V_R(\omega) = V_\circ(\omega)\frac{R}{Z_C(\omega) + R} = V_\circ(\omega)\frac{1}{1 - i\frac{1}{\omega RC}} = V_\circ(\omega)\frac{1}{\sqrt{1 + \frac{1}{\omega^2 R^2 C^2}}} e^{i\tan^{-1}\frac{1}{\omega RC}} \; .$$

Thus, the CR circuit is a *high pass filter* since the transmitted amplitude rises from 0 to 1 as the frequency increases from 0 to ∞. Note that the phase changes from $\pi/2$ to 0.