Chapter 6.

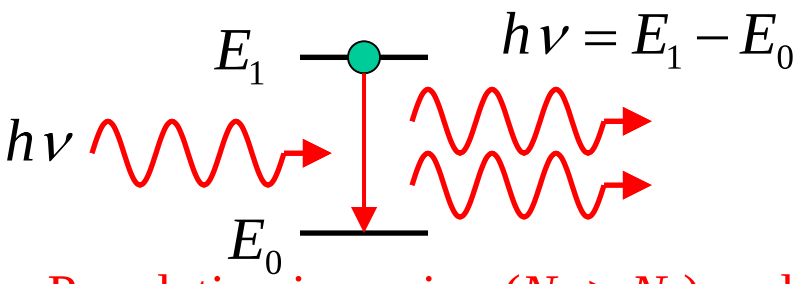
Laser: Theory and Applications

Reading: Sigman, Chapter 6, 7, and 26
Bransden & Joachain, Chapter 15

Laser Basics

Light Amplification by Stimulated Emission of Radiation

Stimulated emission



$h\nu = E_1 - E_0$

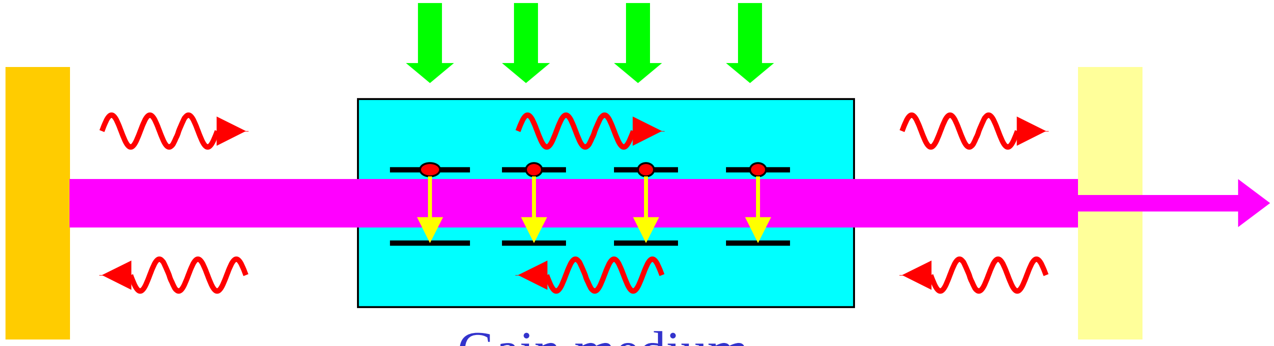
Population inversion ($N_1 > N_0$) ⇒ laser, maser

Transition rate
for stimulated emission

$$\overline{W}_{01} = \frac{4\pi^2}{m^2 c}\left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{I(\omega_{01})}{\omega_{01}^2}\left|\overline{M}_{01}(\omega_{01})\right|^2 \propto \boxed{I(\omega_{01})}$$

Incident light intensity

Pumping (optical, electrical, etc.) for population inversion



Gain medium

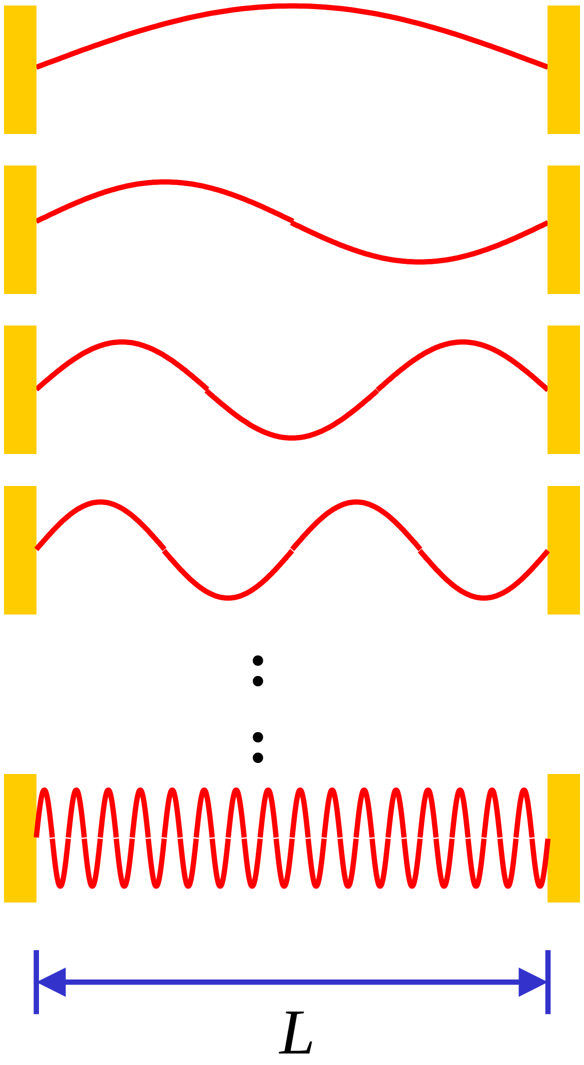
High reflector

Out coupler

Optical cavity

Longitudinal Modes in an Optical Cavity

EM wave in a cavity



Boundary condition:

$$L = \frac{\lambda}{2} m = \frac{c\tau}{2} m = \frac{\pi c}{\omega} m$$

m = 1, 2, 3, …

$$\Rightarrow \quad \lambda = \frac{2L}{m}, \nu = \frac{c}{2L} m, \omega = \frac{\pi c}{L} m$$

Round-trip time of flight: $T = \frac{2L}{c} = m\tau$

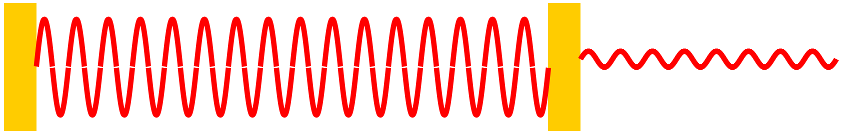
Typical laser cavity: L = 1.5 m, λ = 0.75 μm

$$T = \frac{2L}{c} = \frac{3\,\mathbf{m}}{3\times10^{8}\ \mathbf{m/sec}} = 10^{-8}\ \mathbf{sec} = 10\,\mathbf{nsec}$$

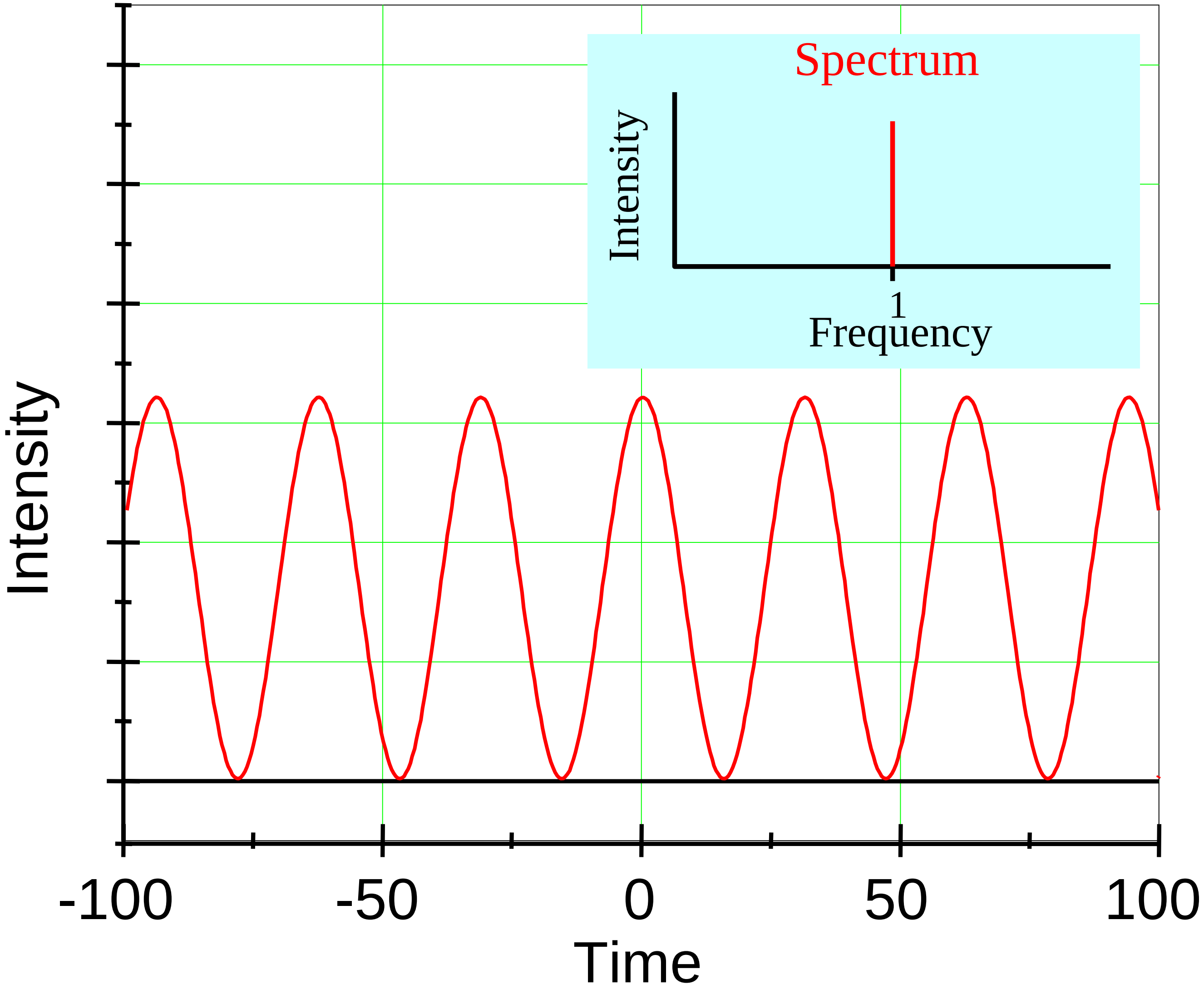
$$\Rightarrow \quad \nu_R = \frac{1}{T} = 10^{8}\ \mathbf{Hz} = 100\,\mathbf{MHz}$$

$$m = \frac{2L}{\lambda} = \frac{3\,\mathbf{m}}{0.75\times10^{-6}\ \mathbf{m}} = 4\times10^{6} = 4\,\mathbf{milion\ !!}$$

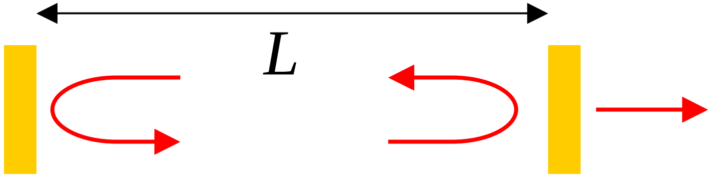
Single mode



$$I(t) = |\cos \omega_0 t|^2 = |\cos t|^2$$



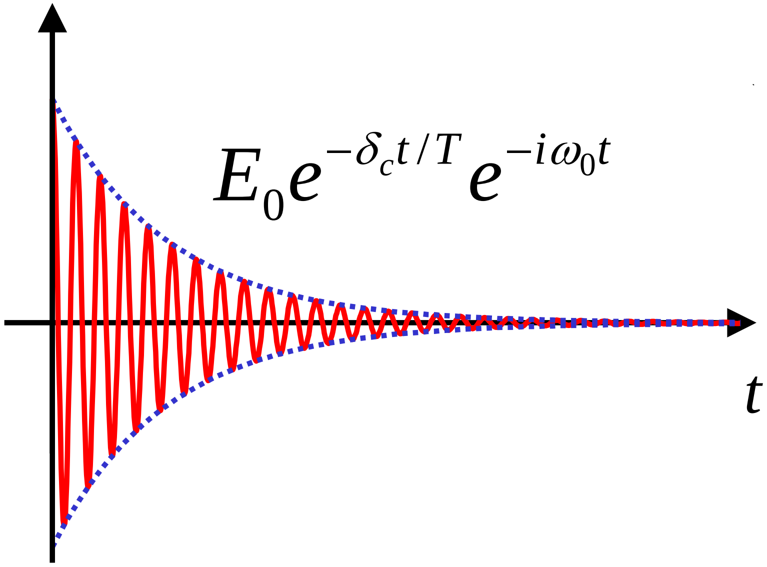
Cavity Quality Factors, Q_c



End mirror
$R \approx 1$

Out coupler
$T = 1 \sim 5\ \%$

Energy loss by reflection, transmission, etc.

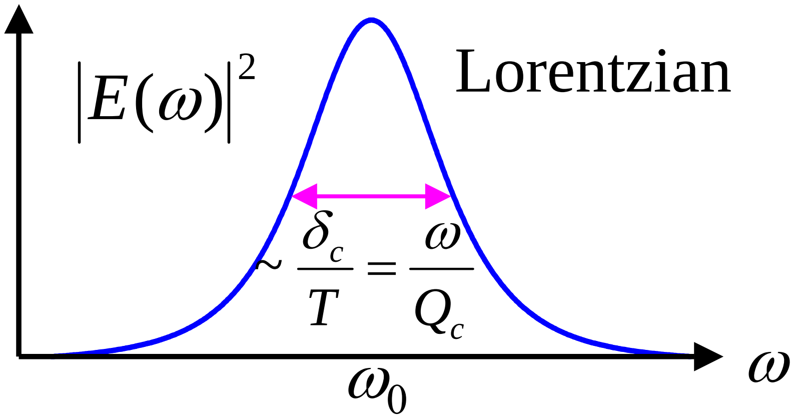


$$E(\omega) = \int_0^\infty E_0 e^{-\delta_c t/T} e^{-i\omega_0 t} dt$$

$$= \frac{E_0}{i(\omega - \omega_0) + \delta_c / T}$$

Emission spectrum

$$|E(\omega)|^2 = \frac{E_0^2}{(\omega - \omega_0)^2 + \delta_c^2 / T^2}$$



Energy of circulating EM wave, $I_{circ}(t)$

$$I_{circ}(t) = I_{circ}(0) \times \mathbf{exp}\left[-\delta_c \boxed{\frac{t}{T}}\right], \qquad T = \frac{2L}{c}$$: round-trip time of flight

Number of round trips in t

$$\Rightarrow \quad I_{circ}(t) = I_{circ}(0) \times \mathbf{exp}\left[-\frac{\omega}{Q_c} t\right]$$

where $Q_c = \dfrac{\omega T}{\delta_c} = \dfrac{4\pi L}{\lambda} \dfrac{1}{\delta_c}$ ⟺

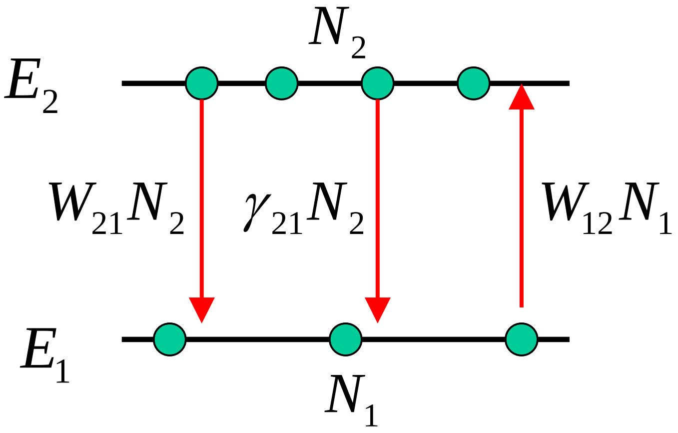
Q-factor of a RLC circuit

$$Q = \frac{\omega}{\Delta\omega} = \frac{\omega L}{R}$$

Typical laser cavity: L = 1 m, λ = 0.8 μm, δ_c = 0.01 (~1% loss/RT)

$$Q_c = \frac{4\pi}{0.8 \times 10^{-6}} \frac{1}{0.01} \approx 1.6 \times 10^{9} \Longrightarrow \Delta\omega \approx 10^{6}\ Hz$$

Two Level Rate Equations and Saturation



$\gamma_{21} = \frac{1}{T_1}$: non-radiative decay rate

Two Level Rate Equation

$$\frac{dN_1(t)}{dt} = -\frac{dN_2(t)}{dt}$$
$$= -W_{12}N_1(t) + (W_{21} + \gamma_{21})N_2(t)$$
$$= -W_{12}\left[N_1(t) - N_2(t)\right] + \gamma_{21}N_2(t)$$

Total number of atoms: $N = N_1(t) + N_2(t) = \text{constant}$

Population difference: $\Delta N(t) = N_1(t) - N_2(t)$

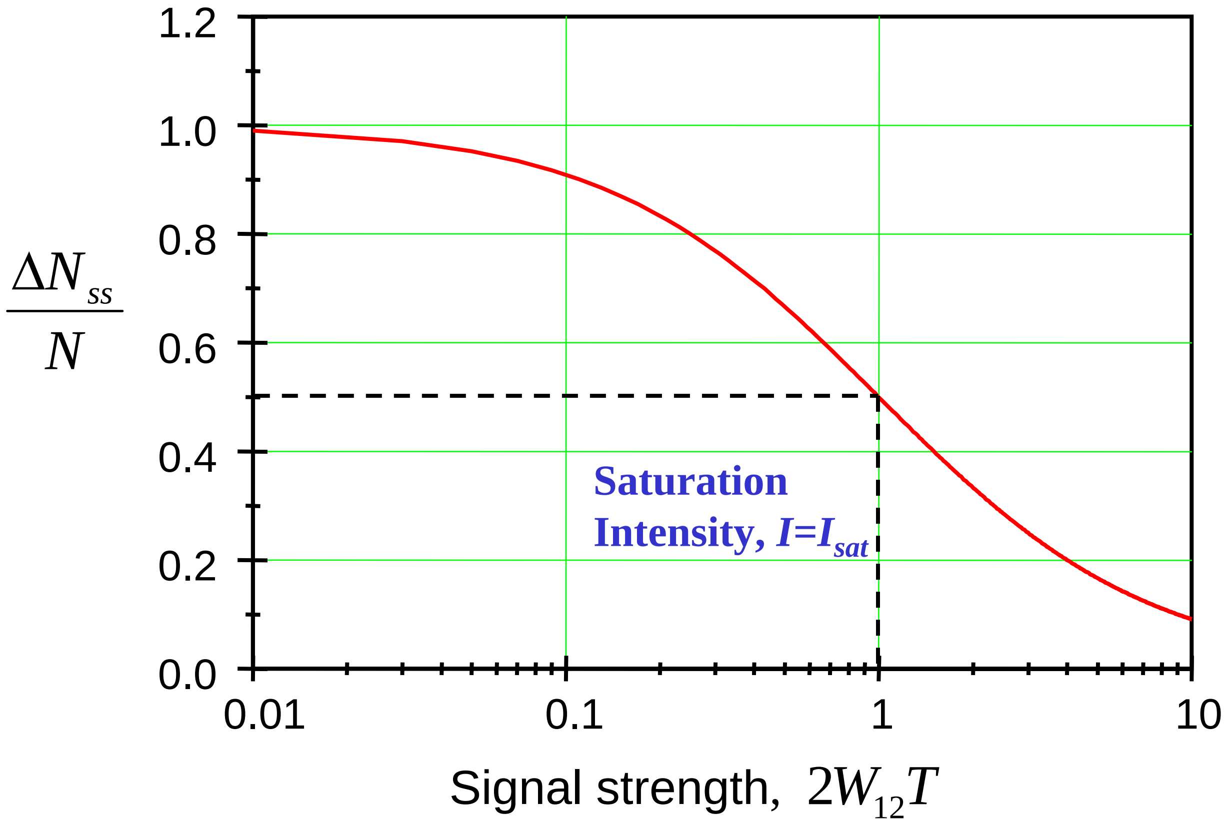
$$\frac{d}{dt}\Delta N(t) = -2W_{12}\left[N_1(t) - N_2(t)\right] + 2\gamma_{21}N_2(t)$$
$$= -2W_{12}\left[N_1(t) - N_2(t)\right] - \gamma_{21}\left[N_1(t) - N_2(t) - N_1(t) - N_2(t)\right]$$

$$\Rightarrow \boxed{\frac{d}{dt}\Delta N(t) = -2W_{12}\Delta N(t) - \frac{\Delta N(t) - N}{T_1}}$$

Steady-State Atomic Response: Saturation

$$\frac{d}{dt}\Delta N(t) = 0 = -2W_{12}\Delta N(t) - \frac{\Delta N(t) - N}{T_1}$$

➡ $$\Delta N = \Delta N_{ss} = \frac{N}{1+2W_{12}T}$$ ➡ $$\boxed{\frac{\Delta N_{ss}}{N} = \frac{1}{1+W_{12}/W_{sat}}}, \quad W_{sat} \equiv \frac{1}{2T_1}$$



Gain coefficient
in laser materials

➡ $\alpha_m \propto \Delta N$

$$\boxed{\alpha_m(I) = \frac{\alpha_{m0}}{1+I/I_{sat}}}$$

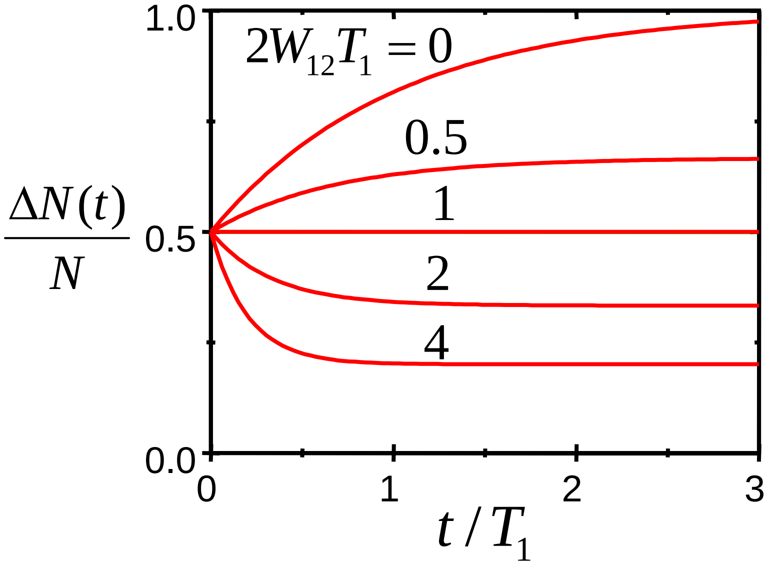
Transient Two-Level Solutions

$$\frac{d}{dt}\Delta N(t) = -2W_{12}\Delta N(t) - \frac{\Delta N(t)-N}{T_1} = -\left(2W_{12}+\frac{1}{T_1}\right)\Delta N(t)+\frac{N}{T_1}$$

⇒ $$\Delta N(t) = \Delta N_{ss} + A\exp\left[-\left(2W_{12}+\frac{1}{T_1}\right)t\right], \quad \Delta N_{ss} = \frac{N}{1+2W_{12}T}$$

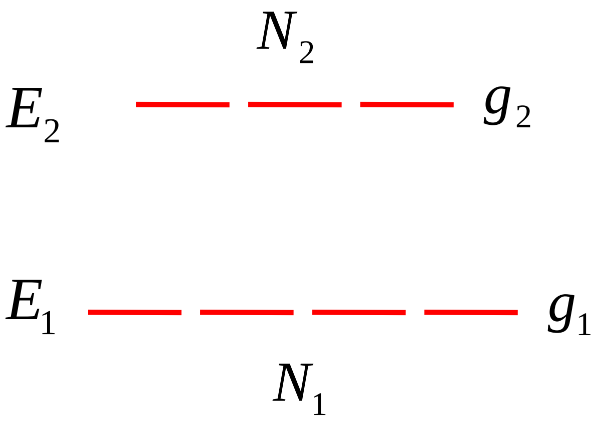
Initial population difference at t = 0: $\Delta N(0) = \Delta N_{ss} + A$

⇒ $$\Delta N(t) = \frac{N}{1+2W_{12}T} + \left[\Delta N(0) - \frac{N}{1+2W_{12}T}\right]\exp\left[-\left(1+2W_{12}T_1\right)\frac{t}{T_1}\right]$$



Transient saturation behavior following sudden turn-on of an applied signal

Two-Level Systems with Degeneracy



Stimulated transition rates: $g_1 W_{12} = g_2 W_{21}$

Rate equation:

$$\frac{dN_1(t)}{dt} = -\frac{dN_2(t)}{dt} = -W_{12} N_1(t) + (W_{21} + \gamma_{21}) N_2(t)$$

Population difference: $\Delta N(t) \equiv \left(\frac{g_2}{g_1} \right) N_1(t) - N_2(t)$

Effective signal-stimulated transition probability: $W_{eff} \equiv \frac{1}{2}\left(W_{12} + W_{21} \right)$

Rate equation: $\frac{d}{dt}\Delta N(t) = -2W_{eff}\Delta N(t) - \frac{\Delta N(t) - N}{T_1}$

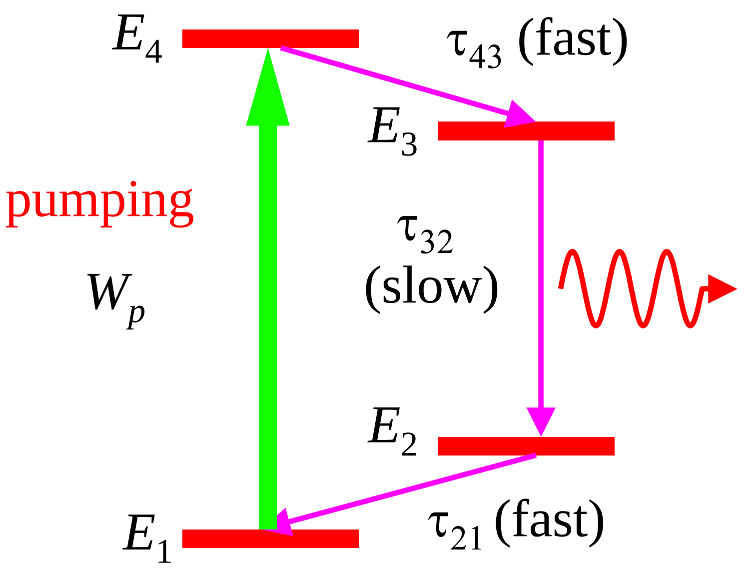
Atomic time constants: *T_1* and *T_2*

T_1 : longitudinal (on-diagonal) relaxation time
population recovery or energy decay time

T_2 : dephasing time, transverse (off-diagonal) relaxation time
time constant for dephasing of coherent macroscopic polarization

Steady-State Laser Pumping and Population Inversion

Four-level pumping analysis



Rate equation for level 4

pump transition probability, $W_{14} = W_{41} = W_p$

$$\frac{dN_4}{dt} = W_p(N_1 - N_4) - (\gamma_{43} + \gamma_{42} + \gamma_{41})N_4$$

$$= W_p(N_1 - N_4) - N_4 / \tau_4,$$

where $\frac{1}{\tau_4} \equiv \gamma_4 = \gamma_{43} + \gamma_{42} + \gamma_{41}$

Steady state population:

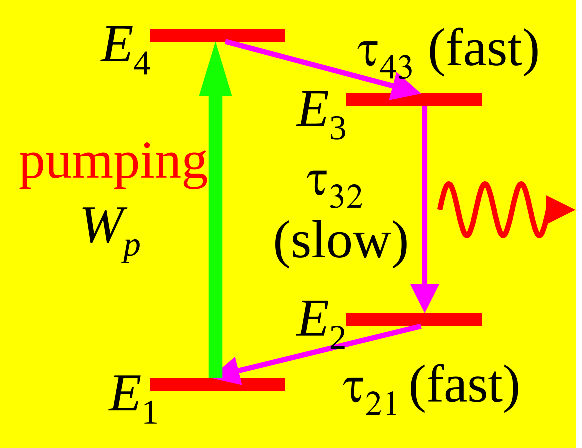
$$N_4 = \frac{W_p \tau_4}{1 + W_p \tau_4} N_1 \approx W_p \tau_4 N_1, \quad \text{if } \boxed{W_p \tau_4} << 1$$

Normalized pumping rate

Rate equations for level 2 and 3

at steady state

$$\frac{dN_3}{dt} = \gamma_{43} N_4 - (\gamma_{32} + \gamma_{31}) N_3 = \frac{N_4}{\tau_{43}} - \frac{N_3}{\tau_3} \Rightarrow N_3 = \frac{\tau_3}{\tau_{43}} N_4$$



In a good laser system, $\tau_3 >> \tau_{43}$ so that $N_3 >> N_4$

$$\frac{dN_2}{dt} = \gamma_{42} N_4 + \gamma_{32} N_3 - \gamma_{21} N_2 = \frac{N_4}{\tau_{42}} + \frac{N_3}{\tau_{32}} - \frac{N_2}{\tau_{21}}$$

$$\Rightarrow N_2 = \left(\frac{\tau_{21}}{\tau_{32}} + \frac{\tau_{43}\tau_{21}}{\tau_{42}\tau_3} \right) N_3 = \beta N_3 \quad \text{where} \quad \beta \equiv \frac{\tau_{21}}{\tau_{32}} + \frac{\tau_{43}\tau_{21}}{\tau_{42}\tau_3}$$

If $\beta < 1$, $N_2 < N_3$: population inversion on the $3 \rightarrow 2$ transition

In a good laser system, $\gamma_{42} \approx 0$ so that the level 4 will relax primarily into the level 3.

$\beta \approx \frac{\tau_{21}}{\tau_{32}}$ ⇒ condition for population inversion $\boxed{\beta \equiv \frac{N_2}{N_3} \approx \frac{\tau_{21}}{\tau_{32}} << 1}$

Fluorescent quantum efficient

The number of fluorescent photons spontaneously emitted on the laser transition divided by the number of pump photons absorbed on the pump transitions when the laser material is below threshold

$$\eta = \frac{\gamma_{43}}{\gamma_4} \times \frac{\gamma_{rad}}{\gamma_3} = \boxed{\frac{\tau_4}{\tau_{43}}} \times \boxed{\frac{\tau_3}{\tau_{rad}}}$$

Fraction of the total atoms excited to level 4 relax directly into the level 3

Fraction of the total decay out of level 3 is purely radiative decay to level2

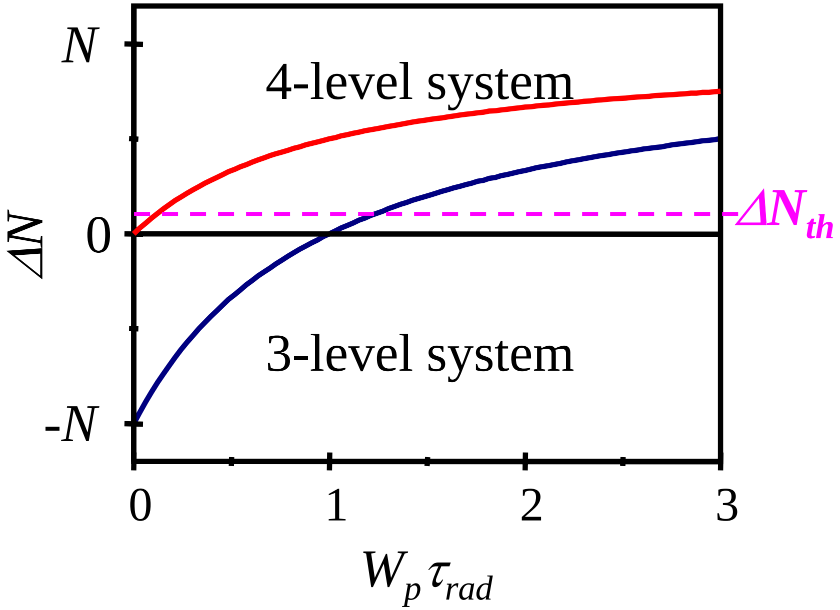
Four level population inversion

$$N = N_1 + N_2 + N_3 + N_4$$

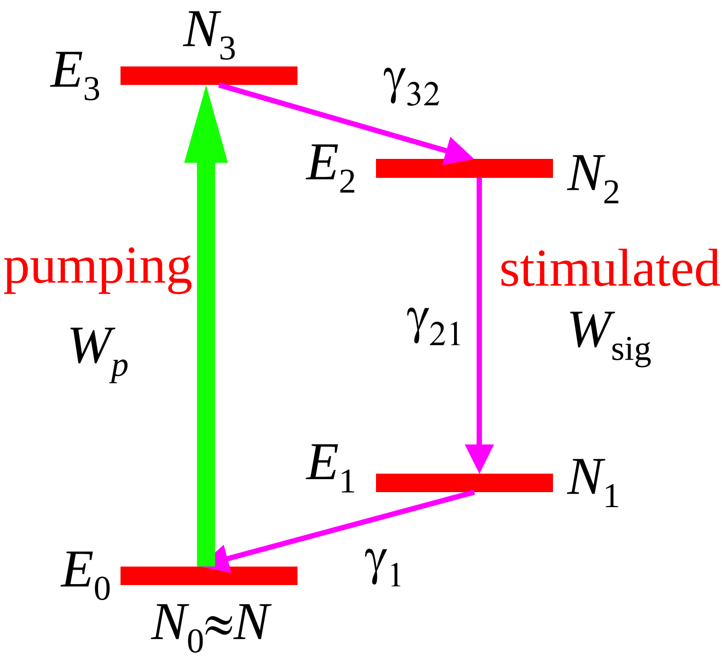
$$\frac{N_3 - N_2}{N} = \frac{(1-\beta)\eta W_p \tau_{rad}}{1 + \left[1 + \beta + 2\tau_{43}/\tau_{rad}\right]\eta W_p \tau_{rad}}$$

In a good laser system, $\tau_{rad} >> \tau_{43}$, $\beta \to 0$.

$$\frac{N_3 - N_2}{N} \approx \frac{(1-\beta)\eta W_p \tau_{rad}}{1 + (1+\beta)\eta W_p \tau_{rad}} \approx \frac{\eta W_p \tau_{rad}}{1 + \eta W_p \tau_{rad}}$$



Laser gain saturation analysis



Pumping transition

$$\left.\frac{dN_3}{dt}\right|_{pump} = W_p(N_0 - N_3) \approx W_p N_0 \approx W_p N$$

Effective pumping rate: $R_p = \boxed{\eta_p} W_p N_0$

quantum efficiency $E_3 \rightarrow E_2$

Rate equations for laser levels 1 and 2: $\gamma_2 = \gamma_{21} + \gamma_{20}$

$$\frac{dN_2}{dt} = R_p - W_{sig}(N_2 - N_1) - \gamma_2 N_2$$

$$\frac{dN_1}{dt} = W_{sig}(N_2 - N_1) + \gamma_{21} N_2 - \gamma_1 N_1$$

$\Rightarrow$

$$N_1 = \frac{W_{sig} + \gamma_{21}}{W_{sig}(\gamma_1 + \gamma_{20}) + \gamma_1\gamma_2} R_p$$

$$N_2 = \frac{W_{sig} + \gamma_1}{W_{sig}(\gamma_1 + \gamma_{20}) + \gamma_1\gamma_2} R_p$$

Gain saturation behavior

$$\Delta N_{21} = N_2 - N_1 = \left(\frac{\gamma_1 - \gamma_{21}}{\gamma_1 \gamma_2}\right) R_p \times \frac{1}{1 + \left[(\gamma_1 + \gamma_{20}) / \gamma_1 \gamma_2\right] W_{sig}}$$

$$= \Delta N_0 \frac{1}{1 + W_{sig} \tau_{eff}}$$

Small signal or unsaturated population inversion

$$\Delta N_0 = \left(\frac{\gamma_1 - \gamma_{21}}{\gamma_1 \gamma_2}\right) R_p = \left(1 - \frac{\tau_1}{\tau_{21}}\right) \times R_p \tau_2$$

Effective recovery time

$$\frac{1}{\tau_{eff}} = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_{20}} \quad \text{or} \quad \tau_{eff} = \tau_2 \left(1 + \frac{\tau_1}{\tau_{20}}\right)$$

If $\gamma_{20} \approx 0, \gamma_2 \approx \gamma_{21}$

$$\Delta N_{21} = R_p (\tau_2 - \tau_1) \times \frac{1}{1 + W_{sig} \tau_2}$$

- Population inversion requires $\tau_{21} > \tau_1$.
- $\Delta N_0 \propto R_p \times \tau_2 (1 - \tau_1 / \tau_2)$
- If $\tau_1 \to 0$, $\tau_{eff} \approx \tau_2$.
- The saturation intensity of the inverted population is independent of R_p.

Wave Propagation in an Atomic Medium

Wave equation in a laser medium

$$\left[\nabla^2 + \omega^2 \mu\varepsilon\left(1 + \boxed{\chi_{at}}\ \boxed{- i\sigma / \omega\varepsilon}\right)\right] E(x,y,z) = 0$$

atomic susceptibility

Ohmic loss

Plane wave approximation

$$\left[\frac{d^2}{dz^2} + \beta^2\left(1 + \chi_{at} - i\sigma / \omega\varepsilon\right)\right] E(z) = 0$$

"Free-space" propagation constant: $\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} n = \frac{2\pi n}{\lambda}$

Propagation factor: $E(z) = E_0 e^{-\Gamma z}$

$$\Gamma^2 = -\beta^2\left(1 + \chi_{at} - i\sigma / \omega\varepsilon\right)$$

⇨ $\Gamma = i\beta\sqrt{1 + \chi_{at} - i\sigma / \omega\varepsilon} = i\beta\sqrt{1 + \chi'(\omega) + i\chi''(\omega) - i\sigma / \omega\varepsilon}$

Usually, $\chi_{at}, -i\sigma/\omega\varepsilon << 1$

$$\Gamma \approx i\beta\left[1+\frac{1}{2}\chi'(\omega)+i\frac{1}{2}\chi''(\omega)-i\frac{\sigma}{2\omega\varepsilon}\right]$$

$$= i\beta + i\frac{1}{2}\beta\chi'(\omega)-\frac{1}{2}\beta\chi''(\omega)+\frac{\sigma}{2\varepsilon v}$$

$$= i\beta + i\frac{1}{2}\Delta\beta_m(\omega)-\alpha_m(\omega)+\alpha_0$$

$$\chi_{at}(\omega)=\frac{a}{\omega^2-\omega_a^2+i\Gamma\omega}$$

$$= a\,\frac{\left(\omega^2-\omega_a^2\right)+i\Gamma\omega}{\left(\omega^2-\omega_a^2\right)^2+\Gamma^2\omega^2}$$

$$\chi'(\omega) = \boxed{\frac{a\left(\omega^2-\omega_a^2\right)}{\left(\omega^2-\omega_a^2\right)^2+\Gamma^2\omega^2}}$$

$$+i\,\boxed{\frac{a\gamma\omega}{\left(\omega^2-\omega_a^2\right)^2+\gamma^2\omega^2}}$$

$$\chi''(\omega)$$

Propagation of a +z traveling wave

$$E(z,t)=\mathrm{Re}\,E_0\exp\{i\omega t-i[\beta+\boxed{\Delta\beta_m(\omega)]z}+\boxed{[\alpha_m(\omega)-\alpha_0]z}\}$$

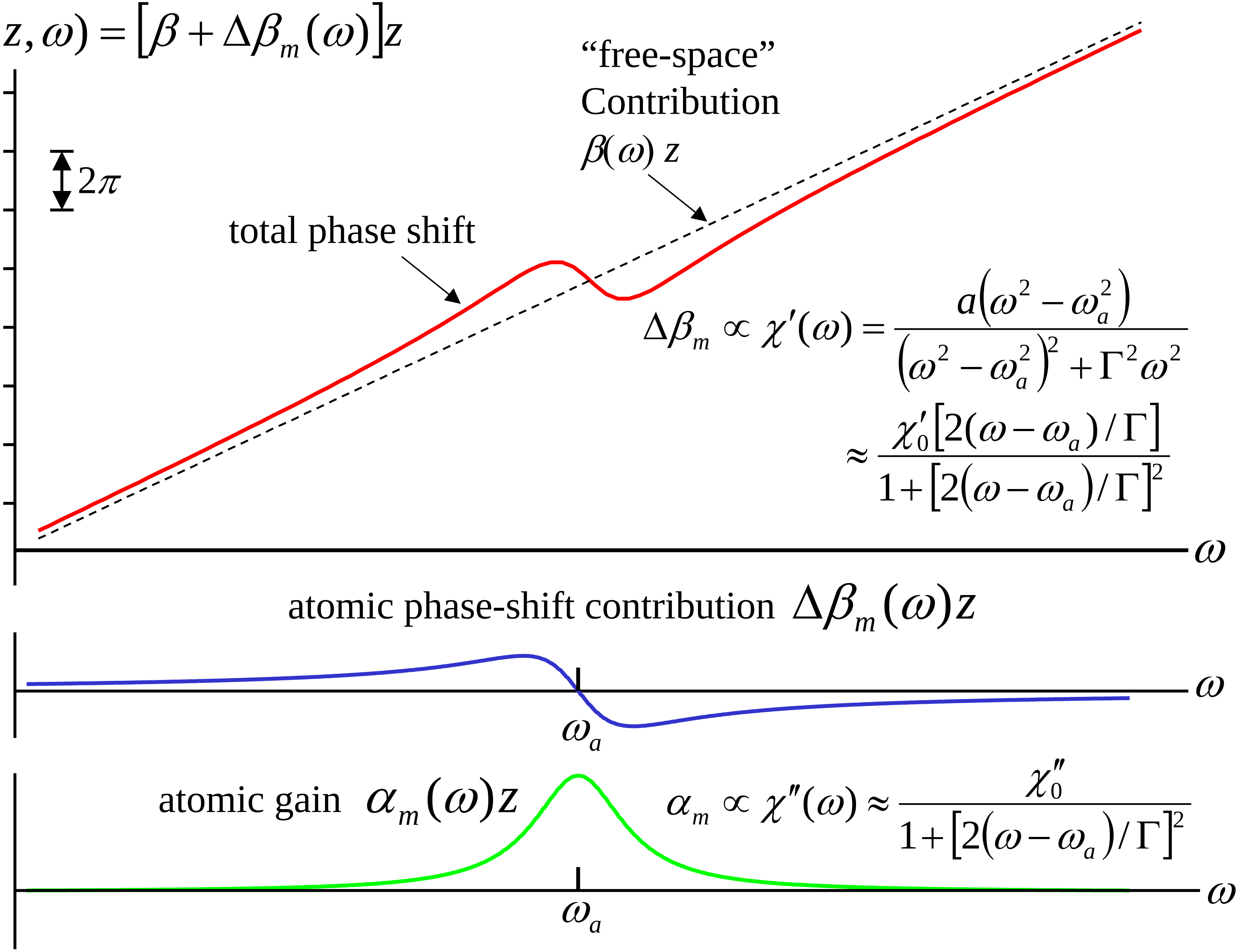
Phase shift
by atomic transition

Gain by atomic transiton
+ ohmic loss

The effects of ohmic losses and atomic transition are included.

Propagation factors

$$\phi_{tot}(z,\omega) = \left[\beta + \Delta\beta_m(\omega)\right]z$$



“free-space” Contribution

$\beta(\omega)\ z$

total phase shift

$$\Delta\beta_m \propto \chi'(\omega) = \frac{a\left(\omega^2 - \omega_a^2\right)}{\left(\omega^2 - \omega_a^2\right)^2 + \Gamma^2\omega^2}$$

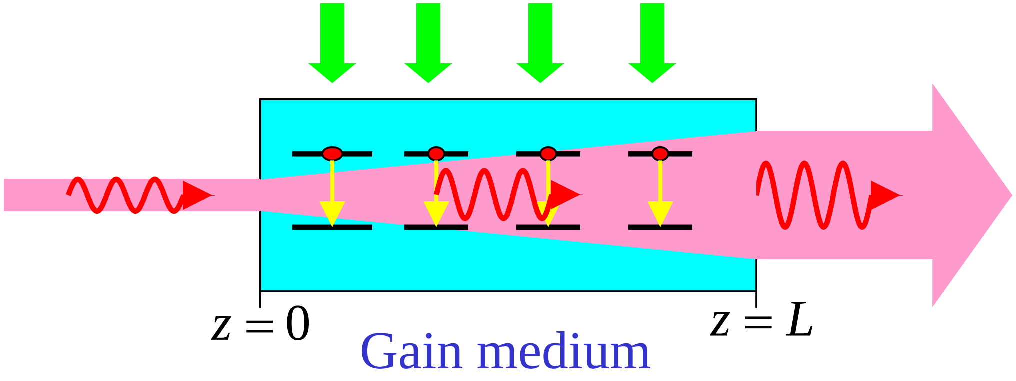
$$\approx \frac{\chi_0'\left[2(\omega - \omega_a)/\Gamma\right]}{1 + \left[2\left(\omega - \omega_a\right)/\Gamma\right]^2}$$

atomic phase-shift contribution $\Delta\beta_m(\omega)z$

atomic gain $\alpha_m(\omega)z$

$$\alpha_m \propto \chi''(\omega) \approx \frac{\chi_0''}{1 + \left[2\left(\omega - \omega_a\right)/\Gamma\right]^2}$$

Single-Pass Laser Amplification



Laser gain formulas

Complex amplitude gain: $g(\omega)\equiv\frac{E(L)}{E(0)}=\exp\{\boxed{-i[\beta+\Delta\beta_m(\omega)]L}+\boxed{[\alpha_m(\omega)-\alpha_0]L}\}$

Total phase shift

amplitude
gain or loss

Power or intensity gain: $G(\omega)\equiv\frac{I(L)}{I(0)}=|g(\omega)|^2=\exp\{2[\boxed{\alpha_m(\omega)}-\alpha_0]L\}$

$$=\frac{1}{2}\beta\chi''(\omega)$$

Midband value

Lorenzian transition line shape: $\chi''(\omega)=\frac{\boxed{\chi_0''}}{1+[2(\omega-\omega_a)/\Delta\omega_a]^2}$

Power gain: $G(\omega) = \exp\left\{ \frac{\omega L \chi_0''}{v} \times \frac{1}{1 + \left[2(\omega - \omega_a)/\Delta\omega_a \right]^2} \right\}$

Power gain in decibels (dB):

$$G_{dB}(\omega) \equiv 10 \log_{10} G(\omega) = 4.34 \ln G(\omega) = \frac{4.34 \omega_a L}{v} \chi''(\omega)$$

Amplification bandwidth and gain narrowing

3-dB amplifier bandwidth: $\Delta\omega_{3dB} = \Delta\omega_a \sqrt{\frac{3}{G_{dB}(\omega_a) - 3}}$

Amplifier phase shift

Total phase shift: $\phi_{tot}(z, \omega) = \left[\beta + \Delta\beta_m(\omega) \right] L = \frac{\omega L}{v} + \frac{\beta L}{2} \chi'(\omega)$

Atomic transition phase shift:

$$\Delta\beta_m(\omega) L = \left(2 \frac{\omega - \omega_a}{\Delta\omega_a} \right) \times \alpha_m(\omega) L = \frac{G_{dB}(\omega_a)}{20 \log_{10} e} \times \frac{2(\omega - \omega_a)/\Delta\omega_a}{1 + \left[2(\omega - \omega_a)/\Delta\omega_a \right]^2}$$

Saturation Intensities in Laser Materials

Saturation of the population difference

Traveling wave: $$\frac{dI}{dz} = 2\alpha_m I = \Delta N \boxed{\sigma} I$$

Stimulated transition cross-section

Population difference: $$\Delta N = \Delta N_0 \times \frac{1}{1+W\tau_{eff}} = \Delta N_0 \times \frac{1}{1+I/I_{sat}}$$

$$\frac{1}{I(z)}\frac{dI(z)}{dz} = 2\alpha_m(z) = \frac{2\alpha_{m0}}{1+I(z)/I_{sat}}$$ where $2\alpha_{m0} \equiv \Delta N_0 \sigma$

⇒ $$\int_{I=I_{in}}^{I=I_{out}} \left[\frac{1}{I} + \frac{1}{I_{sat}}\right] dI = 2\alpha_{m0} \int_0^L dz$$

⇒ $$\ln\left(\frac{I_{out}}{I_{in}}\right) + \frac{I_{out}-I_{in}}{I_{sat}} = 2\alpha_{m0} L = \ln G_0$$ where $\boxed{G_0 \equiv \exp(2\alpha_{m0} L)}$

unsaturated power gain

Overall power gain:

$$G \equiv \frac{I_{out}}{I_{in}} = G_0 \times \exp\left[-\frac{I_{out} - I_{in}}{I_{sat}} \right]$$

$$= G_0 \times \exp\left[-\frac{(G-1)I_{in}}{I_{sat}} \right] = G_0 \times \exp\left[-\frac{(G-1)I_{out}}{GI_{sat}} \right]$$

➡ $\dfrac{I_{out}}{I_{sat}} = \dfrac{1}{G-1} \ln\left(\dfrac{G_0}{G} \right)$ and $\dfrac{I_{out}}{I_{sat}} = \dfrac{G}{G-1} \ln\left(\dfrac{G_0}{G} \right)$

Power extraction and available power

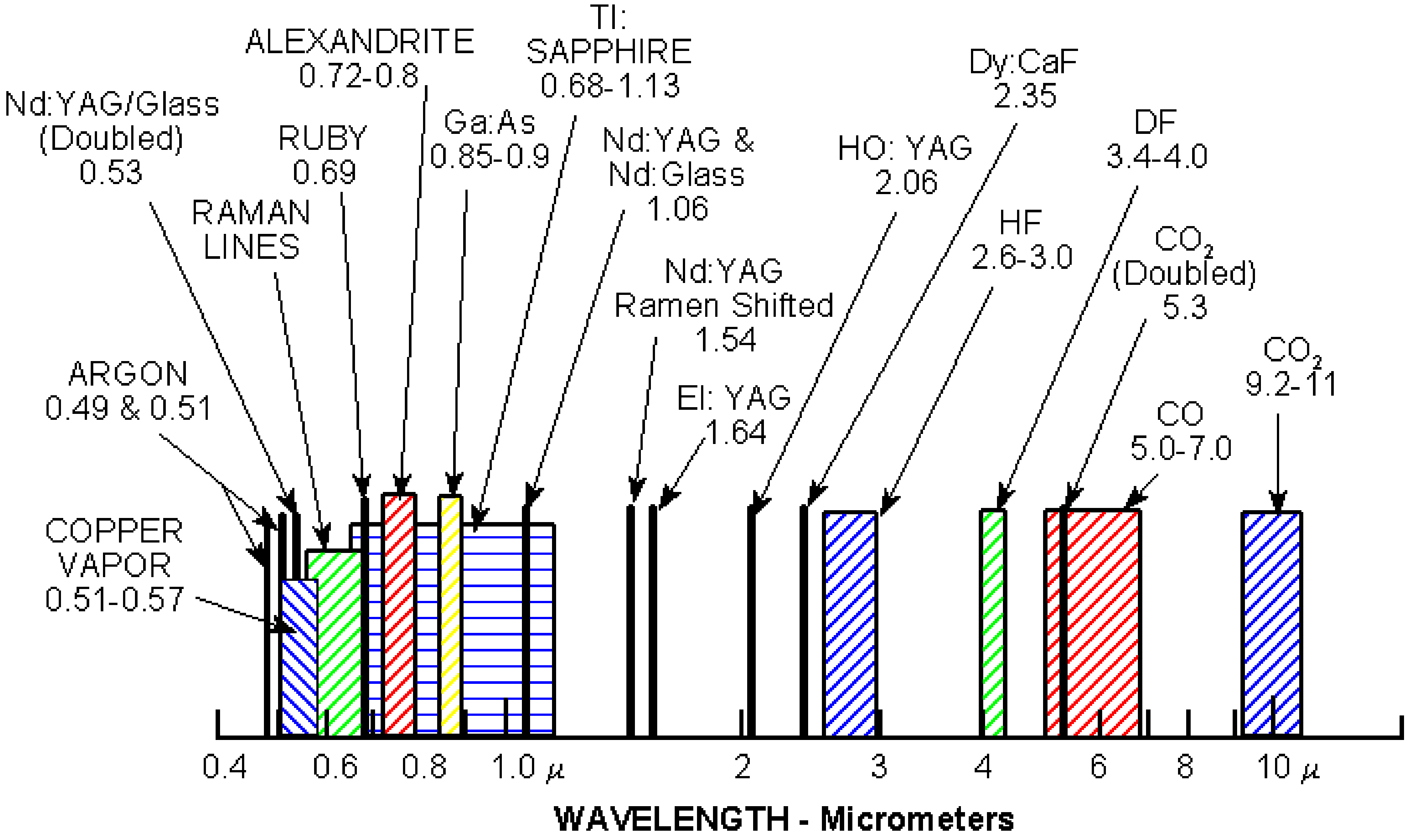
$$I_{extr} \equiv I_{out} - I_{in} = I_{sat} \times \ln\left(\frac{G_0}{G} \right)$$

$$I_{avail} \equiv \lim_{G \to 1} \left[I_{sat} \times \ln\left(\frac{G_0}{G} \right) \right] = I_{sat} \times \ln(G_0) = 2\alpha_{m0} L \times I_{sat}$$

Specific Laser Systems

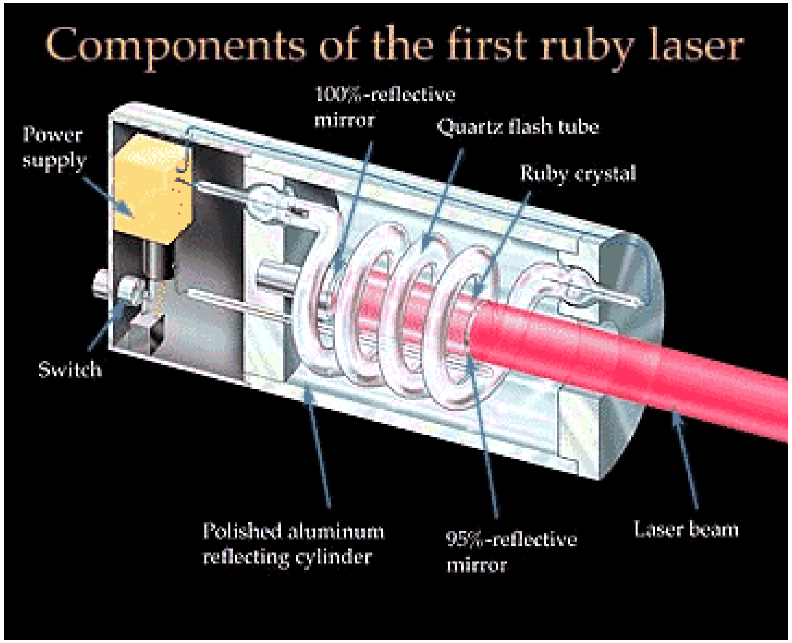
Laser media:

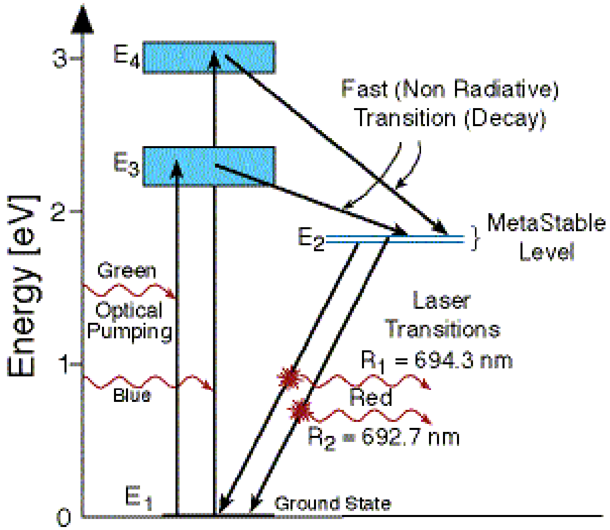
gas, dye, chemical, excimer, solid-state, fiber, semiconductor, free-electron



http://en.wikipedia.org/wiki/Image:Laser_spectral_lines.png

Ruby Laser



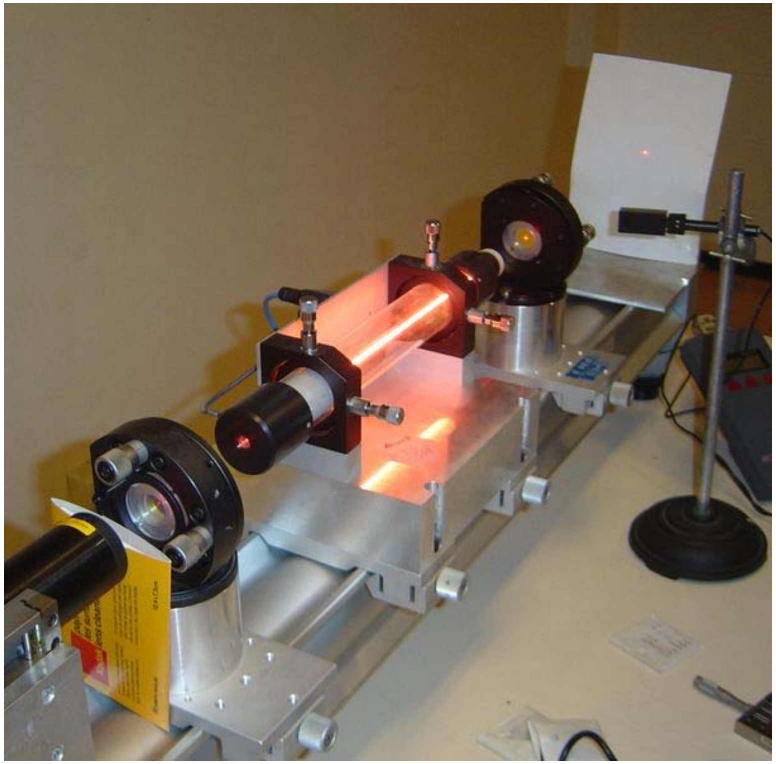


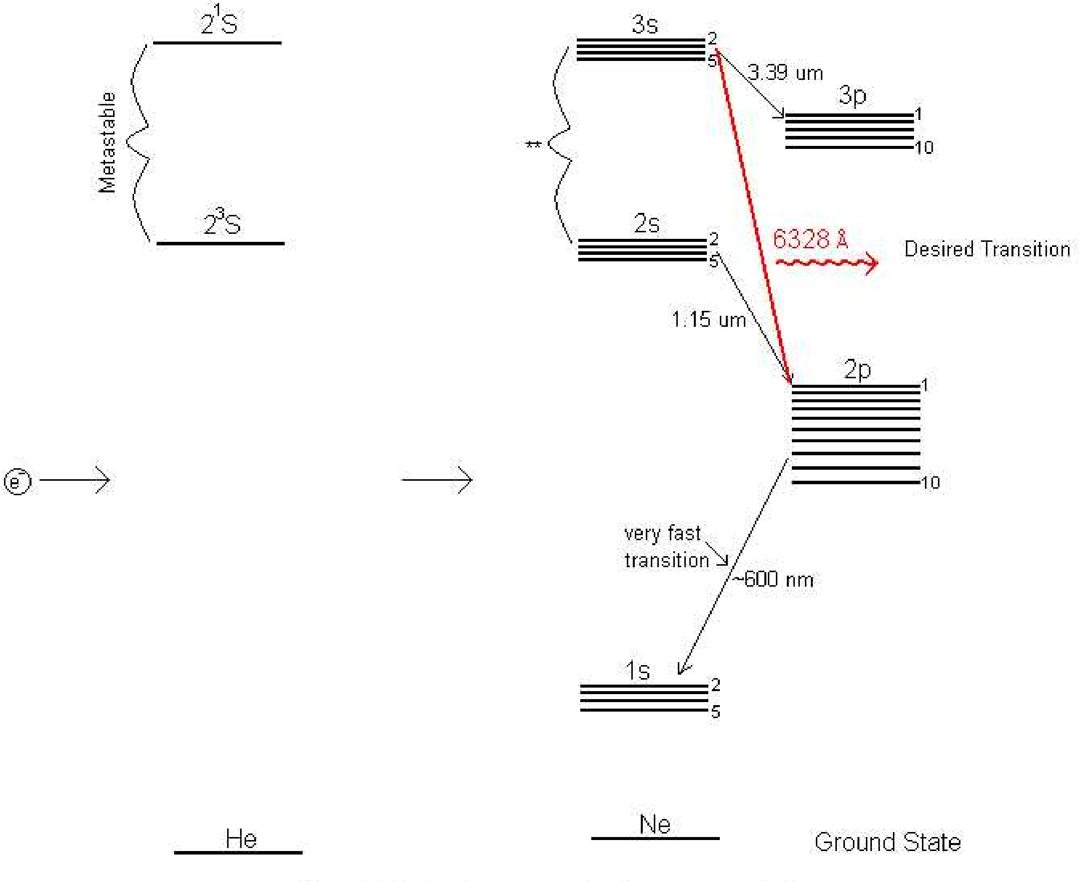
Energy level diagram of Ruby laser

- This system is a **three level laser** with lasing transitions between E_2 and E_1.
- The excitation of the Chromium ions is done by **light pulses** from flash lamps (usually Xenon).
- The **Chromium ions** absorb light at wavelengths around 545 nm (500-600 nm). As a result the ions are transferred to the excited energy level E_3.
- From this level the ions are going down to the **metastable energy level** E_2 in a **non-radiative transition**. The energy released in this non-radiative transition is transferred to the **crystal vibrations** and changed into **heat** that must be removed away from the system.
- The lifetime of the metastable level (E_2) is about 5 msec.

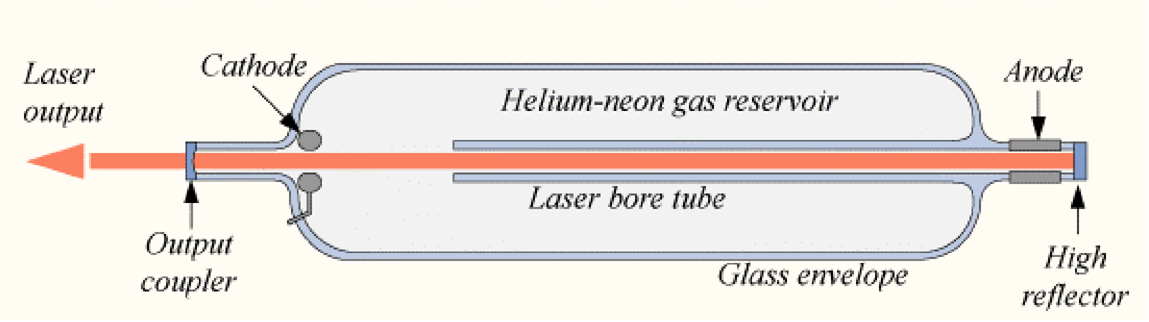
http://web.phys.ksu.edu/vqm/laserweb/Ch-6/C6s2t1p2.htm

He-Ne Laser





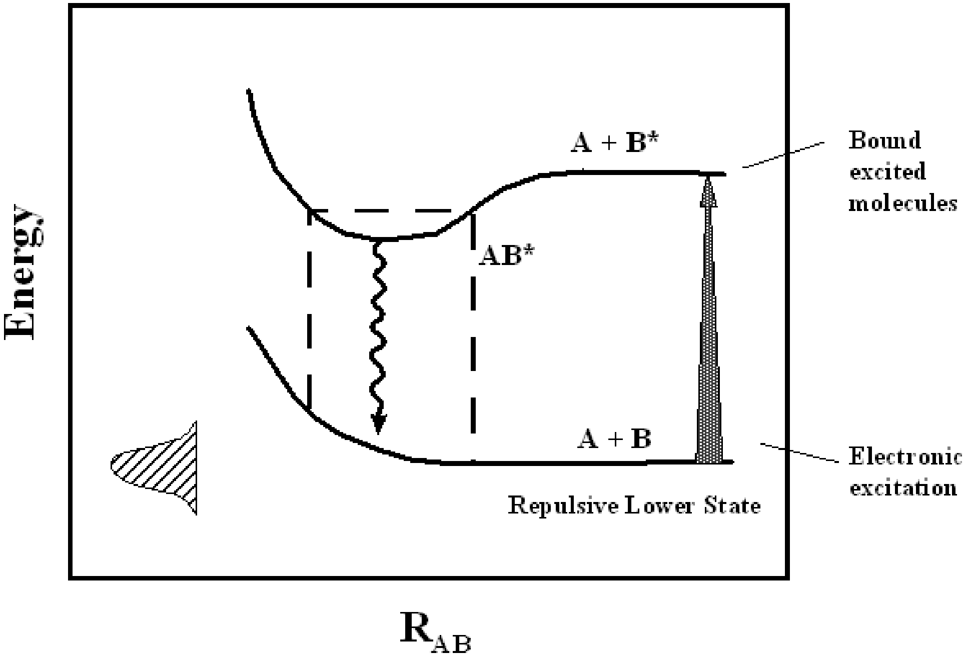
Energy level diagram



Schematics

http://en.wikipedia.org/wiki/Helium-neon_laser

Excimer Lasers



Excimer	Wavelenth
Fe	157 nm
ArF	193 nm
KrF	248 nm
XeCl	308 nm

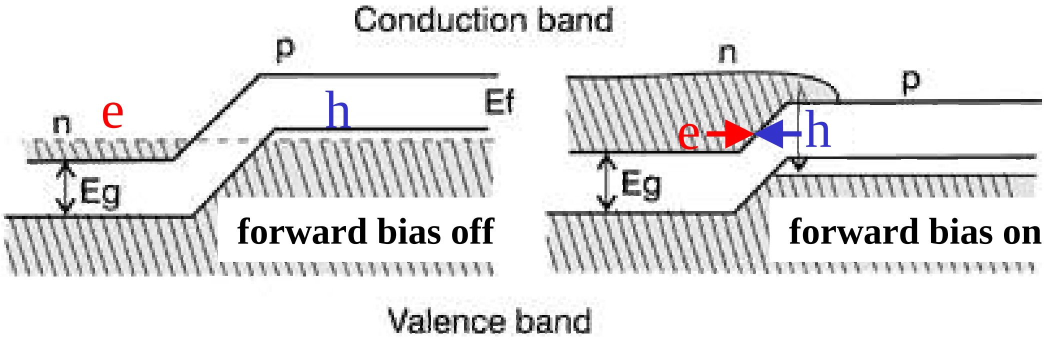
- Gain medium: inert gas (Ar, Kr, Xe etc.) + halide (Cl, F etc.)
- Excited state is induced by an electrical discharge or high-energy electron beams.
- Laser action in an excimer molecule occurs because it has a bound (associative) excited state, but a repulsive (disassociative) ground state.

Applications

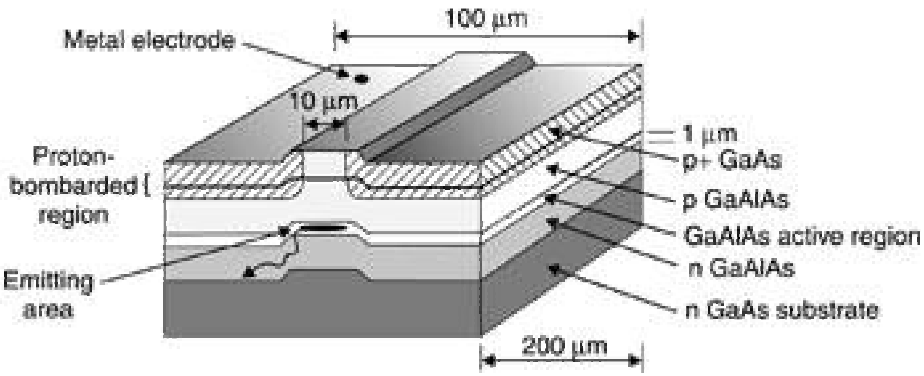
- Marking
- Micromachining
- Laser Ablation
- Laser Annealing
- Surface Structuring
- Laser Vision Correction
- Optical Testing and Inspection
- Pulsed Laser Deposition
- Fiber Bragg Gratings

http://tftlcd.khu.ac.kr/research/poly-Si/chapter4.html

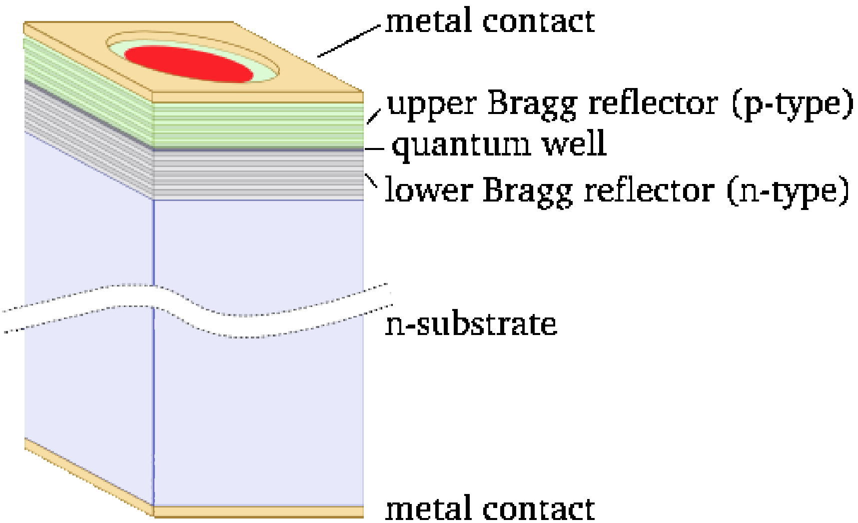
Semiconductor Lasers – laser diodes



Band structure near a semiconductor p-n junction



Diode laser structure



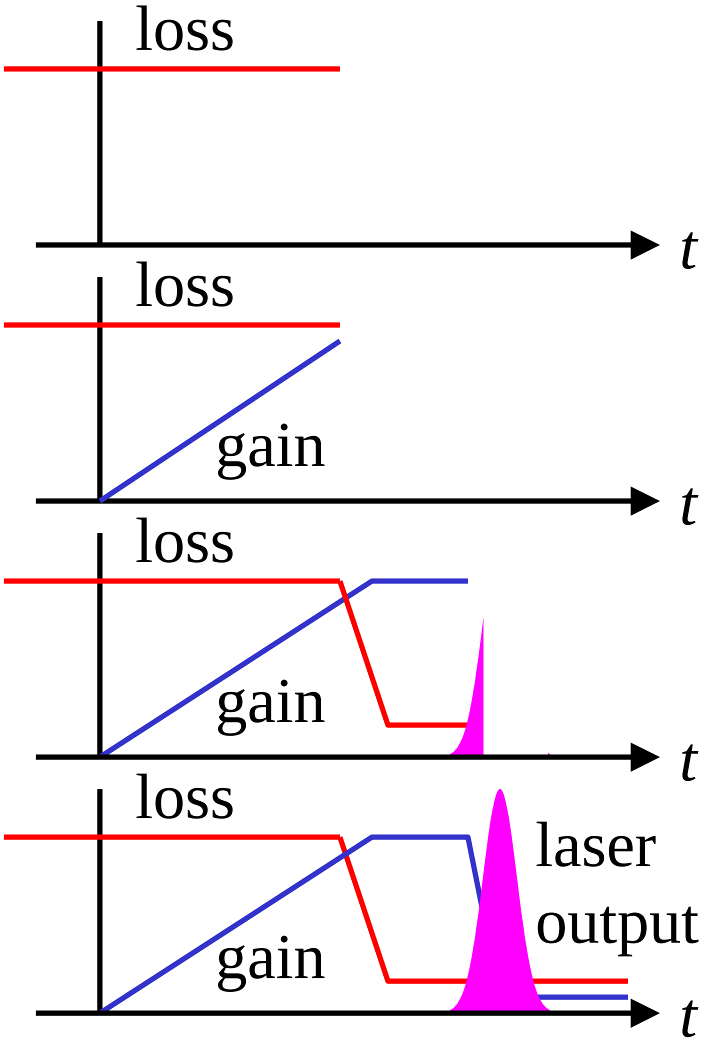
Vertical cavity surface emitting lasers (VCSEL) structure

en.wikipedia.org/wiki/Laser_diode, www.mtmi.vu.lt/pfk/funkc_dariniai/diod/led.htm

Laser Q-Switching

Q-switched laser output:

short and intense burst of laser output dumping all the accumulated population inversion in a single short laser pulse (~10 ns)



High initial cavity loss

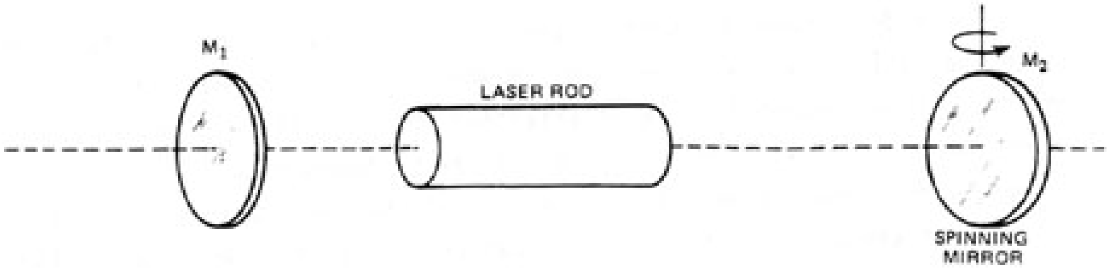
Pumping process builds up a large population inversion

Cavity loss is suddenly “switched” to low value

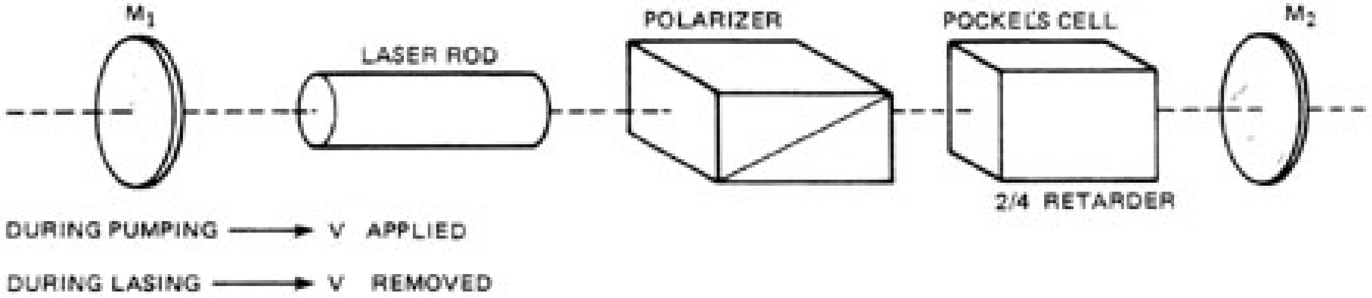
“giant pulse” laser action takes place

Laser Q-switching techniques

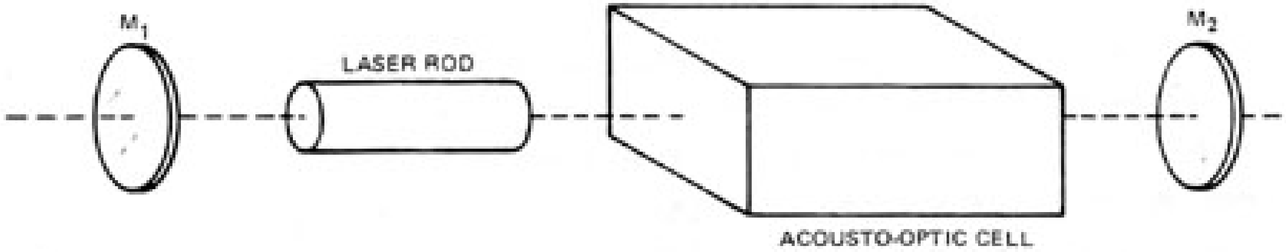
Rotating mirror



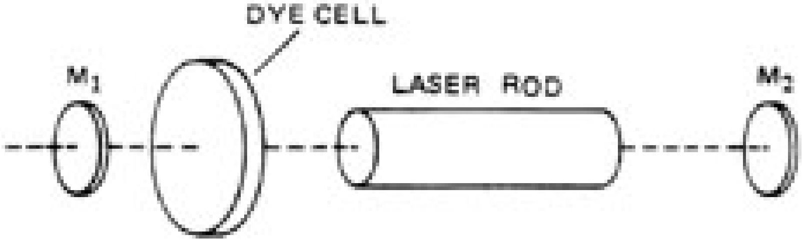
Electro-optic



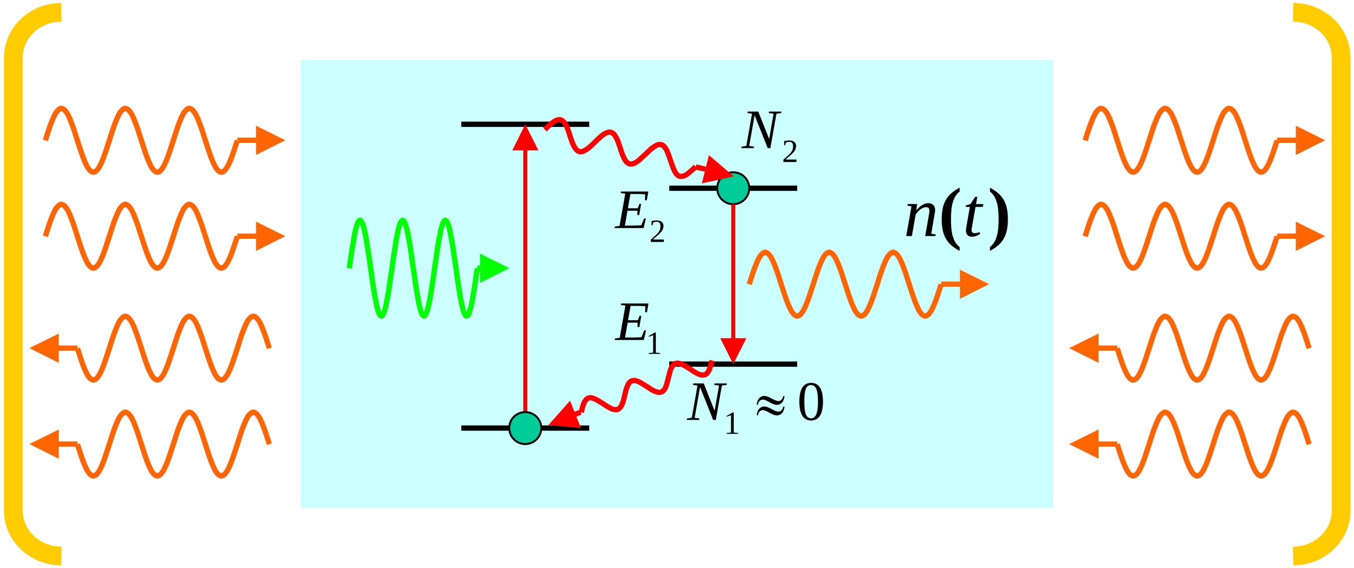
Acousto-optic



Saturable absorber



Active Q-switching: rate-equation analysis



Coupled rate equations

$$\frac{dn(t)}{dt} = KN(t)n(t) - \gamma_c n(t) \qquad \frac{dN(t)}{dt} = R_p - \gamma_2 N(t) - Kn(t)N(t)$$

$n(t)$: cavity photon number

$\gamma_c = 1/\tau_c$: total cavity decay rate

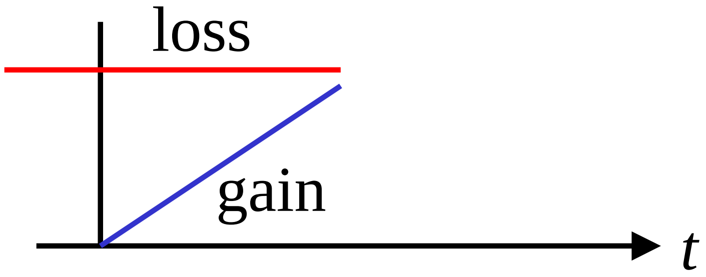
$N(t) = N_2 - N_1$: inverted population difference

R_p : pumping rate

γ_2 : decay rate for N

K : coupling coefficient between photons and atoms

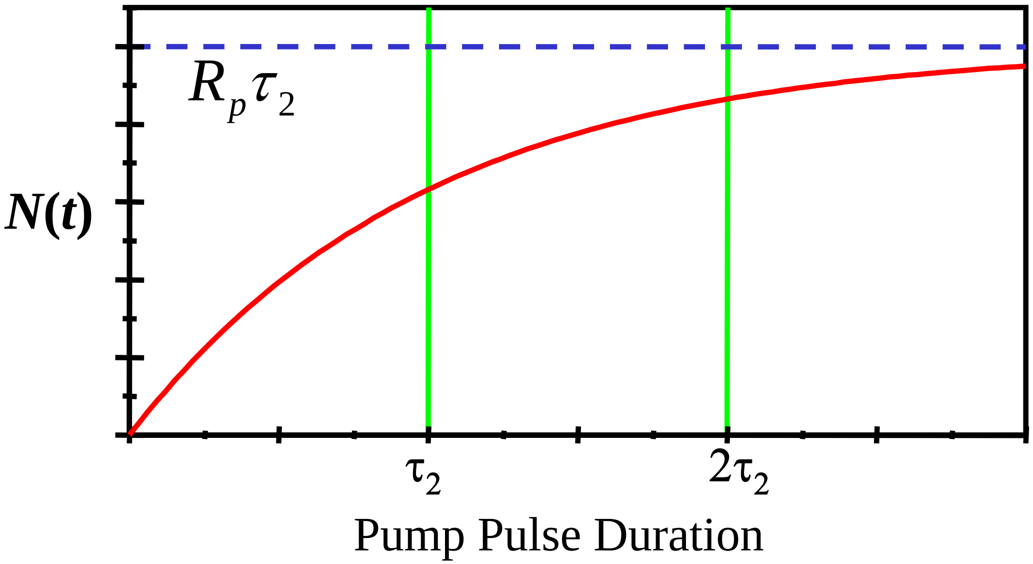
Pumping interval, and population build-up



Pumping process builds up a large population inversion

$n(t)$ = 0 and a pump pulse with constant intensity R_p is turned on at t = 0.

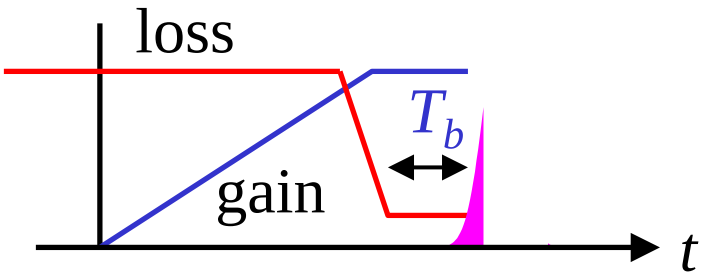
$$\frac{dN(t)}{dt} = R_p - N(t)/\tau_2 \quad \Rightarrow \quad N(t) = R_p\tau_2\left[1 - \mathbf{exp}(-t/\tau_2)\right]$$



Energy storage during the pumping interval for a fixed pumping rate

Typical solid state lasers: $\tau_2 \approx 200\ \mu s$

Pulse build-up time, T_b



Cavity loss is suddenly "switched" to low value

T_b (~100 ns) << τ_2 (~100 μs)

Initial inversion ratio: $r \equiv \dfrac{N_i}{N_{th}}$

N_i : population inversion just after switching

N_{th} : threshold inversion just after switching

$$\frac{dn(t)}{dt} = KN(t)n(t) - \frac{1}{\tau_c}n(t) \approx K\left[N_i - N_{th}\right]n(t) \approx \frac{r-1}{\tau_c}n(t)$$

$$\Rightarrow \quad n(t) = n_i \,\mathbf{exp}\left[(r-1)t/\tau_c\right],$$ $n_i \approx 1$: initial spontaneous emission noise excitation

Steady state photon number, n_{ss} :

- Photon number with continuous pumping at a pumping rate r times above its threshold
- Stimulate emission ($KN(t)n(t)$) becomes significant.
- $n_{ss} << n_p$ (photon number at the peak power)

Build-up time, T_b from the initial photon density n_i to a photon density n_{ss}

$$\frac{n_{ss}}{n_i} = \mathbf{exp}\left[\frac{(r-1)T_b}{\tau_c}\right] \quad \Rightarrow \quad T_b = \frac{\tau_c}{r-1}\,\mathbf{ln}\left(\frac{n_{ss}}{n_i}\right)$$

Ratio of initial to final photon numbers:

$$\frac{n_{ss}}{n_i} \approx 10^8 \ \mathbf{to}\ 10^{12} \quad ➡ \quad \pm 20\ \%\ \text{in}\ \mathbf{ln}\left(\frac{n_{ss}}{n_i}\right)$$

Therefore, $T_b \approx \dfrac{(25 \pm 5)}{r-1} \times \dfrac{T}{\delta_c}$ where $\tau_c = \dfrac{T}{\boxed{\delta_c}}$ and $T = \dfrac{2L}{c}$

Fractional power loss per round trip

Examples:

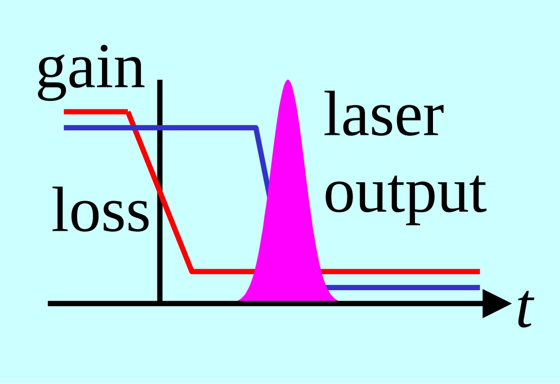
1. Flash-pumped Nd:YAG laser

L = 60 cm (T = 4 ns), R=0.35 (δ_c = ln(1/R)=1.05), r = 3 ⇒ $T_b \approx$ 50 ns

2. cw-pumped Nd:YAG laser or low-gain CO_2 laser

L = 2 m (T = 12 ns), R=0.8 (δ_c = ln(1/R)=0.22), r = 1.5 ⇒ $T_b \approx$ 3 μs

Pulse output interval



$$\frac{dn(t)}{dt} = KN(t)n(t) - \gamma_c n(t) = K\left[N(t) - N_{th}\right]n(t)$$

$$\frac{dN(t)}{dt} = R_p - \gamma_2 N(t) - Kn(t)N(t) \approx -Kn(t)N(t)$$

Initial conditions: $N = N_i = rN_{th}$ and $n = n_i = 1$ at the switching time $t = t_i$

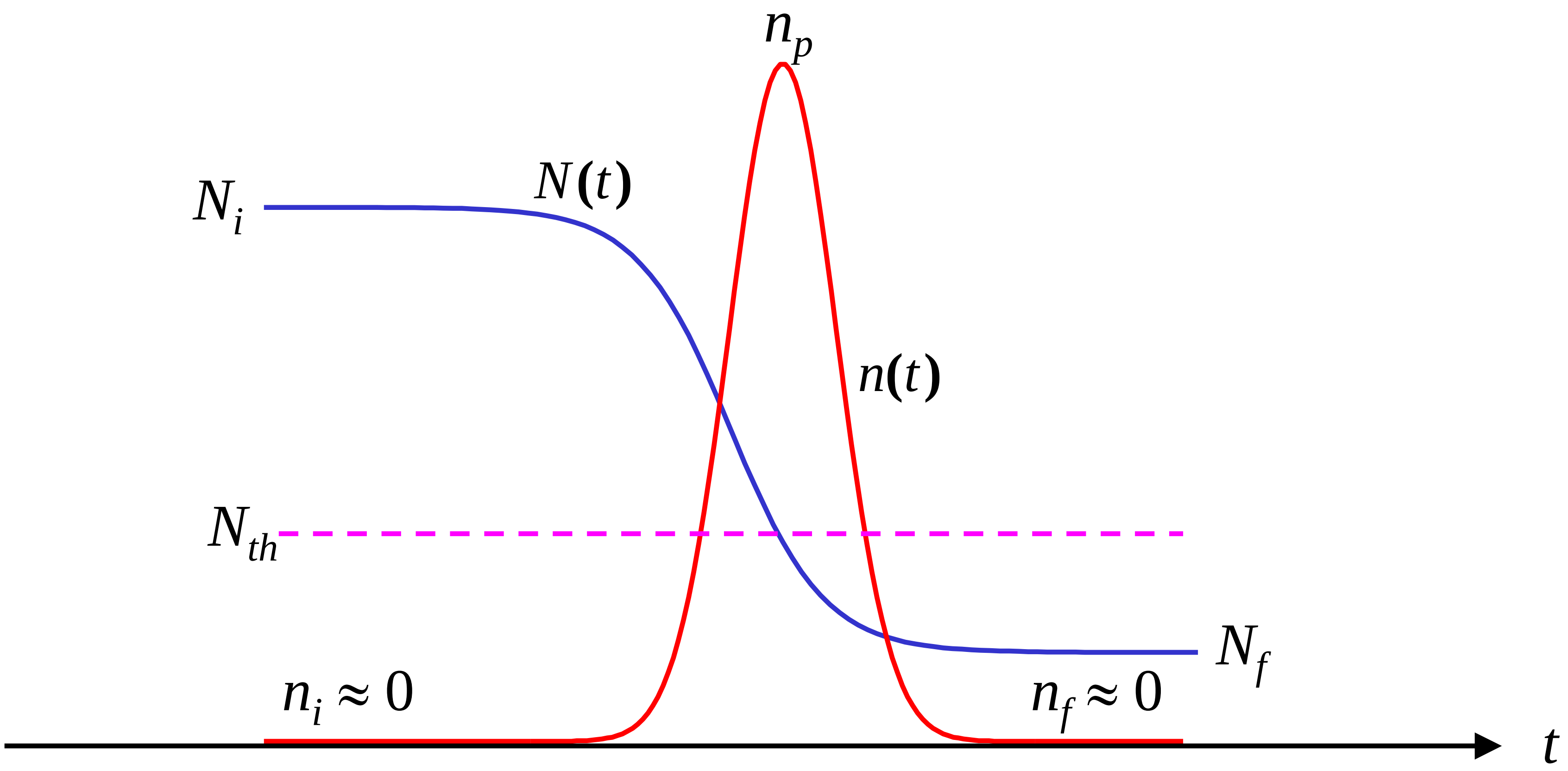
$$\frac{dn}{dN} = \frac{N_{th} - N}{N} \quad \Rightarrow \quad \int_{n_i}^{n(t)} dn = \int_{N_i}^{N(t)} \left(\frac{N_{th}}{N} - 1\right) dN$$

$$\Rightarrow \quad n(t) \approx \int_{N_i}^{N(t)} \left(\frac{N_{th}}{N} - 1\right) dN = N_{th} \ln \frac{N(t)}{N_i} - (N(t) - N_i)$$

$$\Rightarrow \quad n(t) \approx N_i - N(t) - \frac{N_i}{r} \ln \frac{N_i}{N(t)}$$

where $r = \dfrac{N_i}{N_{th}}$

Time evolution of $n(t)$ and $N(t)$ during the pulse output interval



Mode-Locking

Mode-locking is a technique in optics by which a laser can be made to produce ultrashort pulses with the pulse width of the order of subpicoseconds (< 10^{-12}). The basis of the technique is to induce a fixed phase relationship between the modes of the laser's resonant cavity. The laser is then said to be *phase-locked* or *mode-locked*. Interference between these modes causes the laser light to be produced as a train of pulses. Depending on the properties of the laser, these pulses may be of extremely brief duration, as short as a few femtoseconds. - Wikipedia

General description of electric field in a laser cavity

$$E(t) = \sum_m E_m e^{i(\omega_m t + \phi_m)}$$

ϕ_m : initial phase of the *m*th mode

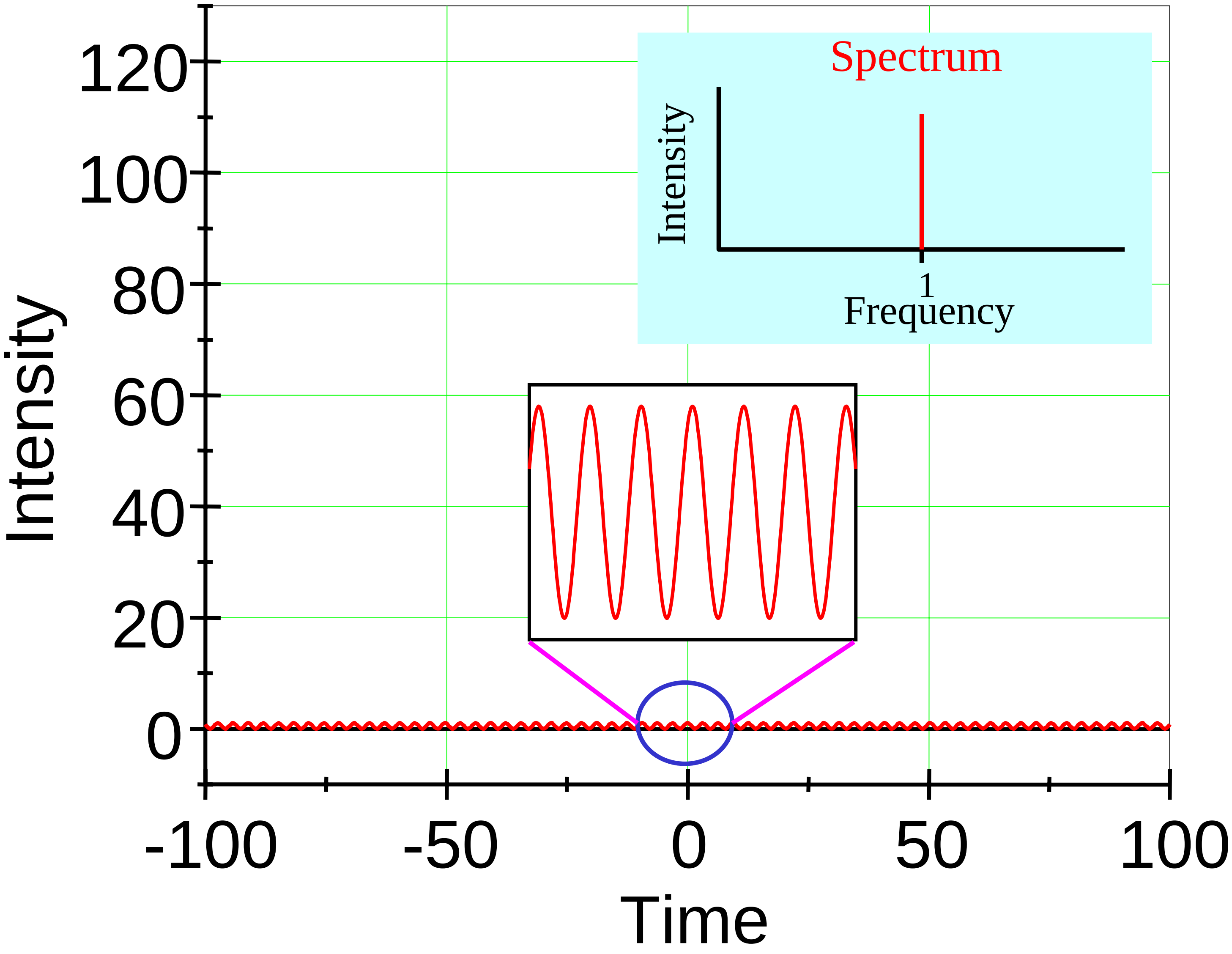
The laser is said to be "mode-locked" when the initial phases are equal.

$$\phi_m = \phi$$

: constant

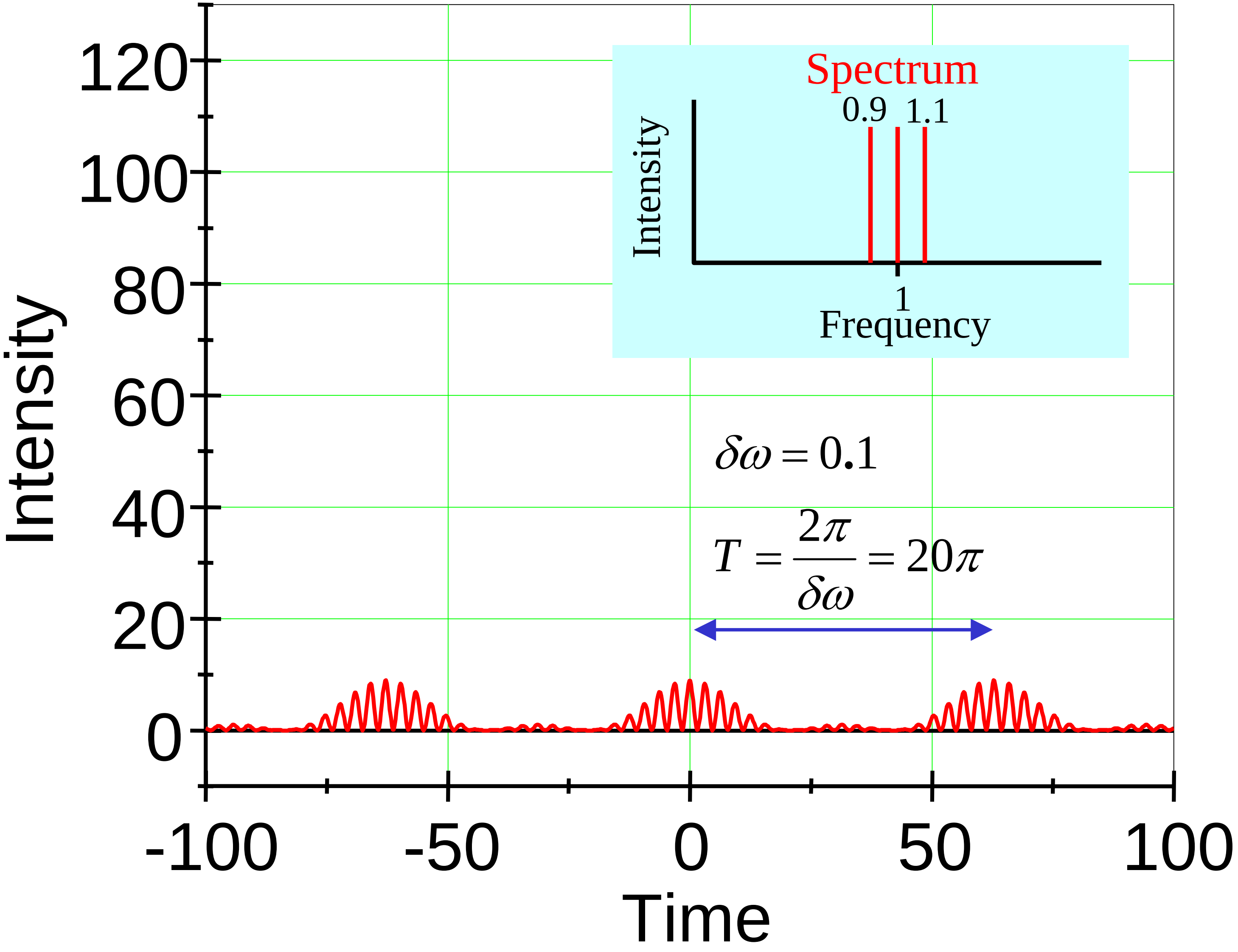
Single mode

$$I(t) = \left|\mathbf{cos}\,\omega_0 t\right|^2 = \left|\mathbf{cos}\,t\right|^2$$



Three modes in phase

$$I(t)=\left|\cos 0.9t+\cos t+\cos 1.1t\right|^2$$

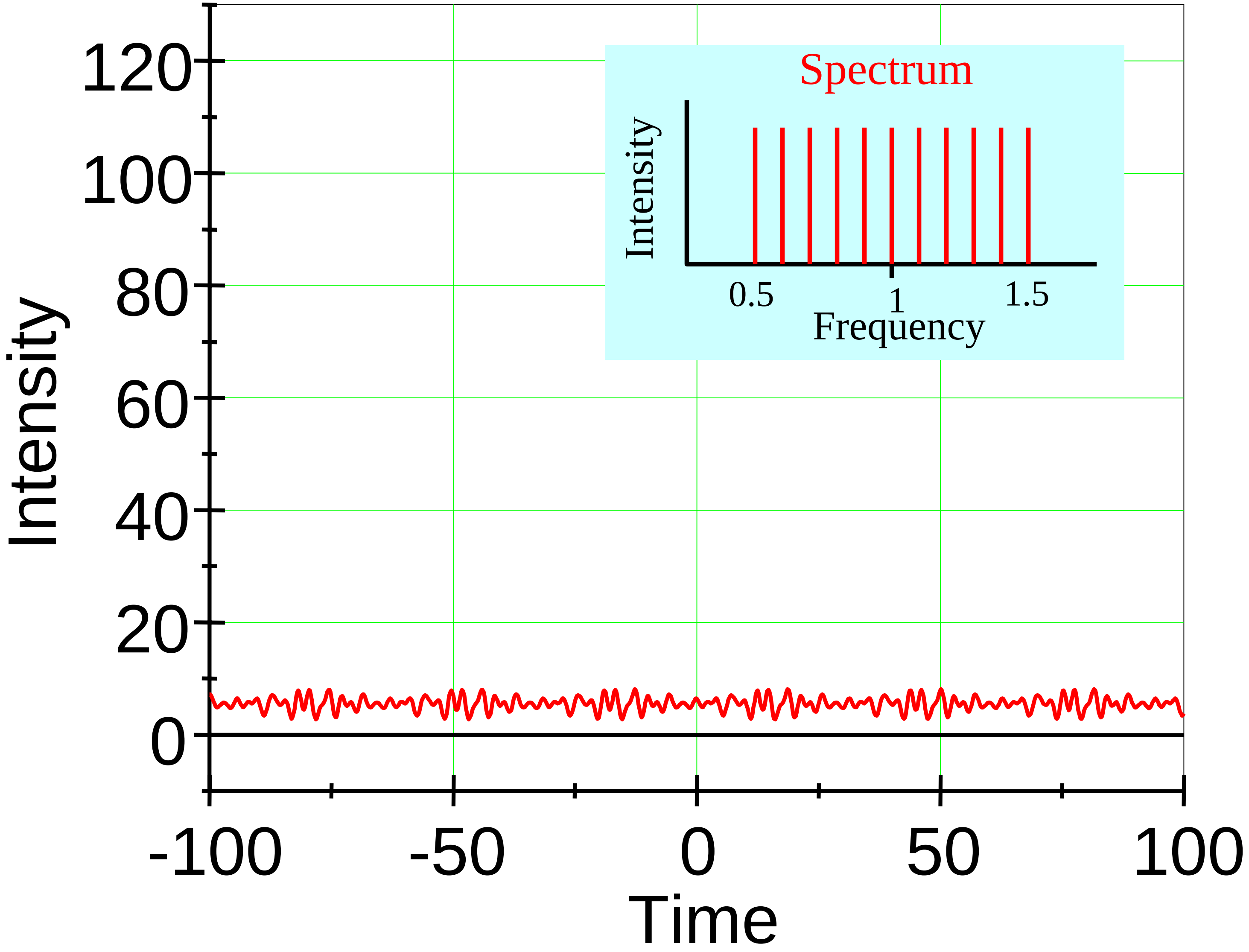


$$\delta\omega=0.1$$

$$T=\frac{2\pi}{\delta\omega}=20\pi$$

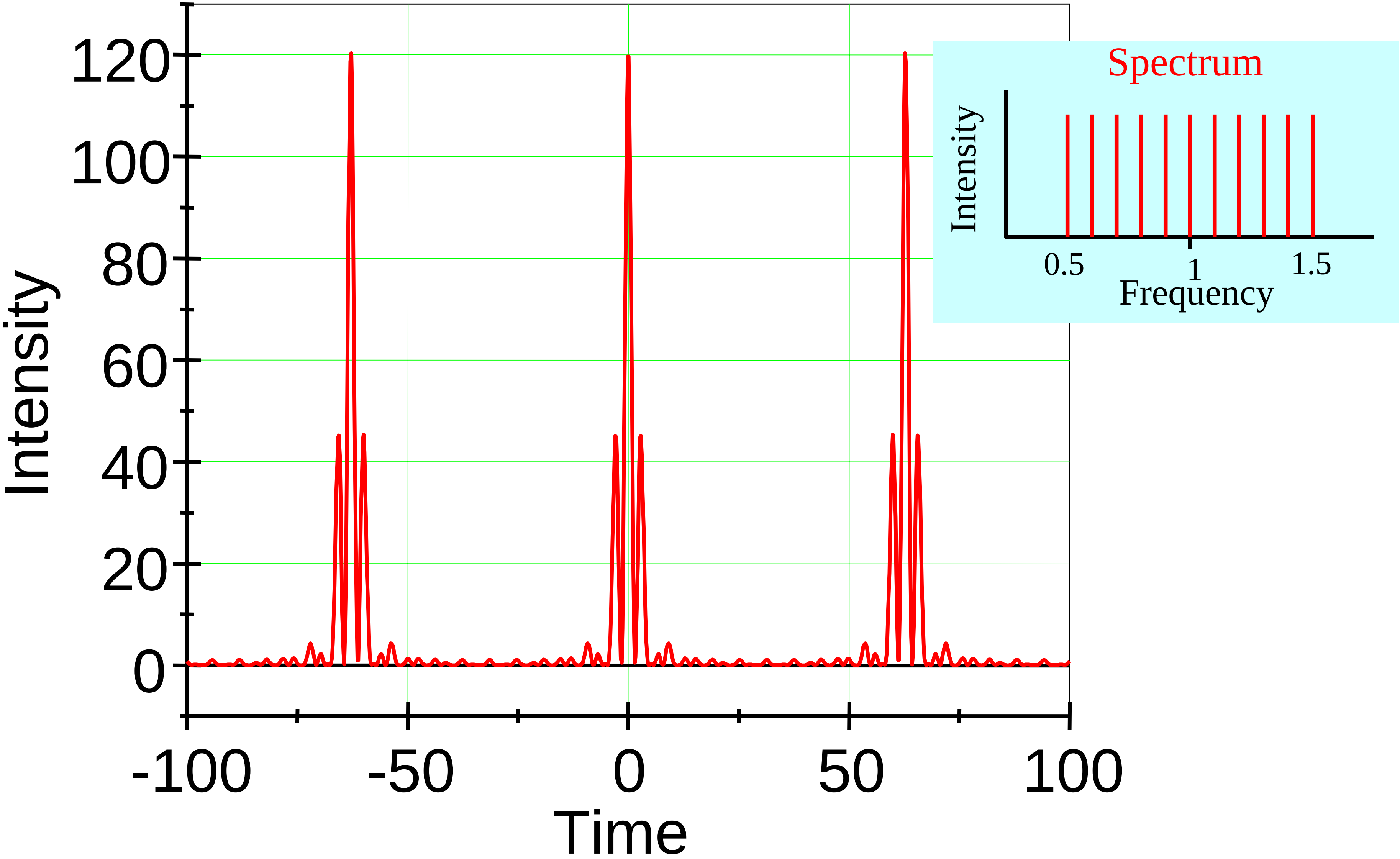
Eleven modes with random phases

$$I(t) = \left|\mathbf{cos}(0.5t + \phi_{0.5}) + \mathbf{cos}(0.6t + \phi_{0.6}) + \mathbf{...}\,\mathbf{cos}(1.0t + \phi_{1.0}) + \mathbf{...} + \mathbf{cos}(1.5t + \phi_{1.5})\right|^2$$



Eleven modes with the same phase, $\phi = 0$

$$I(t) = \left|\mathbf{cos}\,0.5t + \mathbf{cos}\,0.6t + \mathbf{...}\mathbf{cos}\,1.0t + \mathbf{...}\mathbf{cos}\,1.4t + \mathbf{cos}\,1.5t\right|^2$$



In the time domain, EM field is a periodic repetition: $E(t) = E(t+nT)$

$$E(t) = \sum_m E_m e^{i\omega_m t} = \sum_m E_m e^{i\frac{m\pi c}{L}t}$$

$$E(t+nT) = \sum_m E_m e^{i\frac{m\pi c}{L}(t+nT)} = \sum_m E_m e^{i\frac{m\pi c}{L}\left(t+\frac{2nL}{c}\right)} = \sum_m E_m e^{i\frac{m\pi c}{L}t} \overset{=1}{\boxed{e^{2\pi mni}}} = E(t)$$

Spectrum for one period: $E(\omega) = \frac{1}{T}\int_0^T E(t)e^{i\omega t}dt$ Note $E_m = E(\omega_m)$

Normalized spectrum for *N* periods:

$$E^N(\omega) = \frac{1}{NT}\int_0^{NT} E(t)e^{i\omega t}dt = \frac{1}{NT}\sum_{n=0}^{N-1}\int_{nT}^{(n+1)T} E(t)e^{i\omega t}dt$$

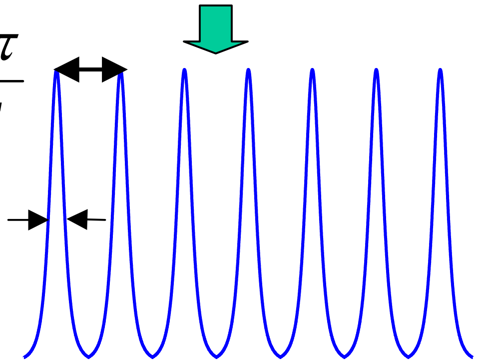
$$= \frac{1}{NT}\sum_{n=0}^{N-1}\int_0^T E(t'+nT)e^{i\omega(t'+nT)}dt' = \frac{1}{N}\sum_{n=0}^{N-1} e^{i\omega nT}\,\frac{1}{T}\int_0^T E(t')e^{i\omega t'}dt'$$

$$= \left(\frac{1}{N}\sum_{n=0}^{N-1} e^{i\omega nT}\right)E(\omega) = \frac{1}{N}\frac{1-e^{i\omega NT}}{1-e^{i\omega T}}E(\omega)$$

Normalized power spectrum for *N* periods:

$$I^N(\omega) = \left|E^N(\omega)\right|^2 = \frac{1}{N^2}\left|\frac{1-e^{i\omega NT}}{1-e^{i\omega T}}\right|^2 \left|E(\omega)\right|^2$$

$$= \boxed{\frac{1}{N^2}\frac{\sin^2(N\omega T/2)}{\sin^2(\omega T/2)}} I(\omega)$$



$$\delta\omega = \frac{2\pi}{T}$$

$$\delta f \approx \frac{1}{NT}$$

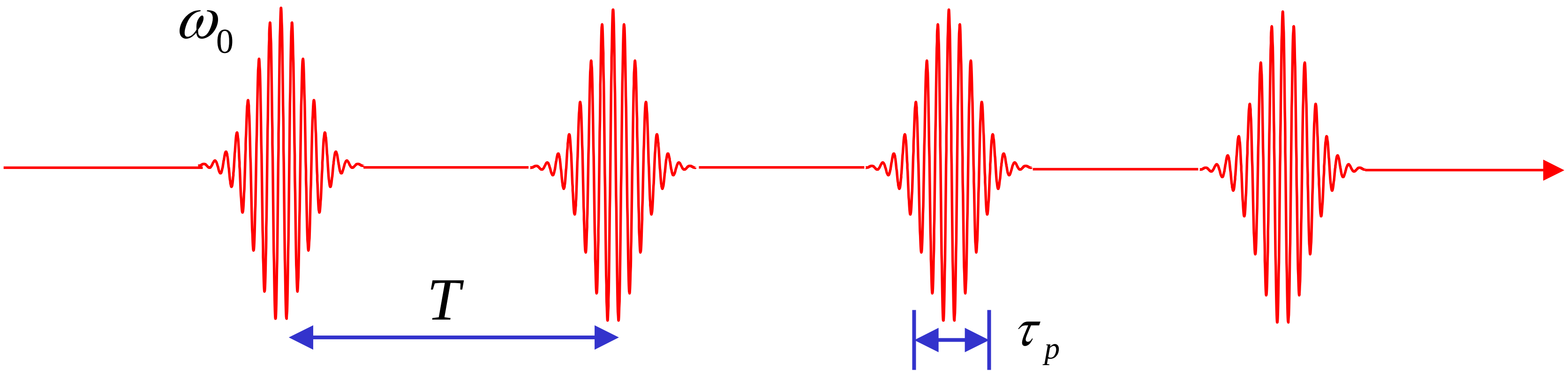
Periodic comb

When *N* goes to infinity, $\frac{\sin^2(N\omega T/2)}{\sin^2(\omega T/2)} \xRightarrow[N\to\infty]{} \sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T}\right)$

Dirac comb

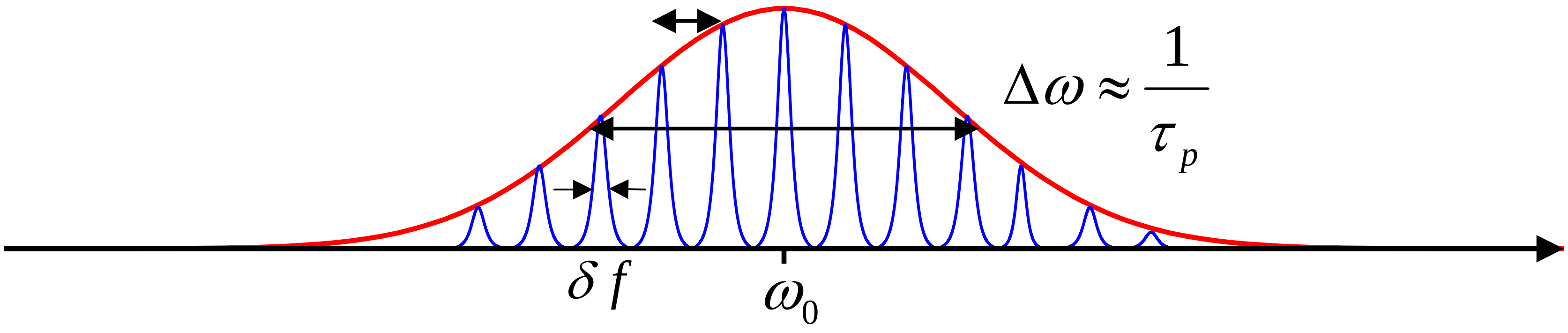
Periodic repetition of EM wave in a laser cavity in the time domain

$$E(t) = E(t + nT)$$



Laser power spectrum $I(\omega)$

$$\delta\omega = \frac{2\pi}{T} = \frac{\pi c}{L}$$



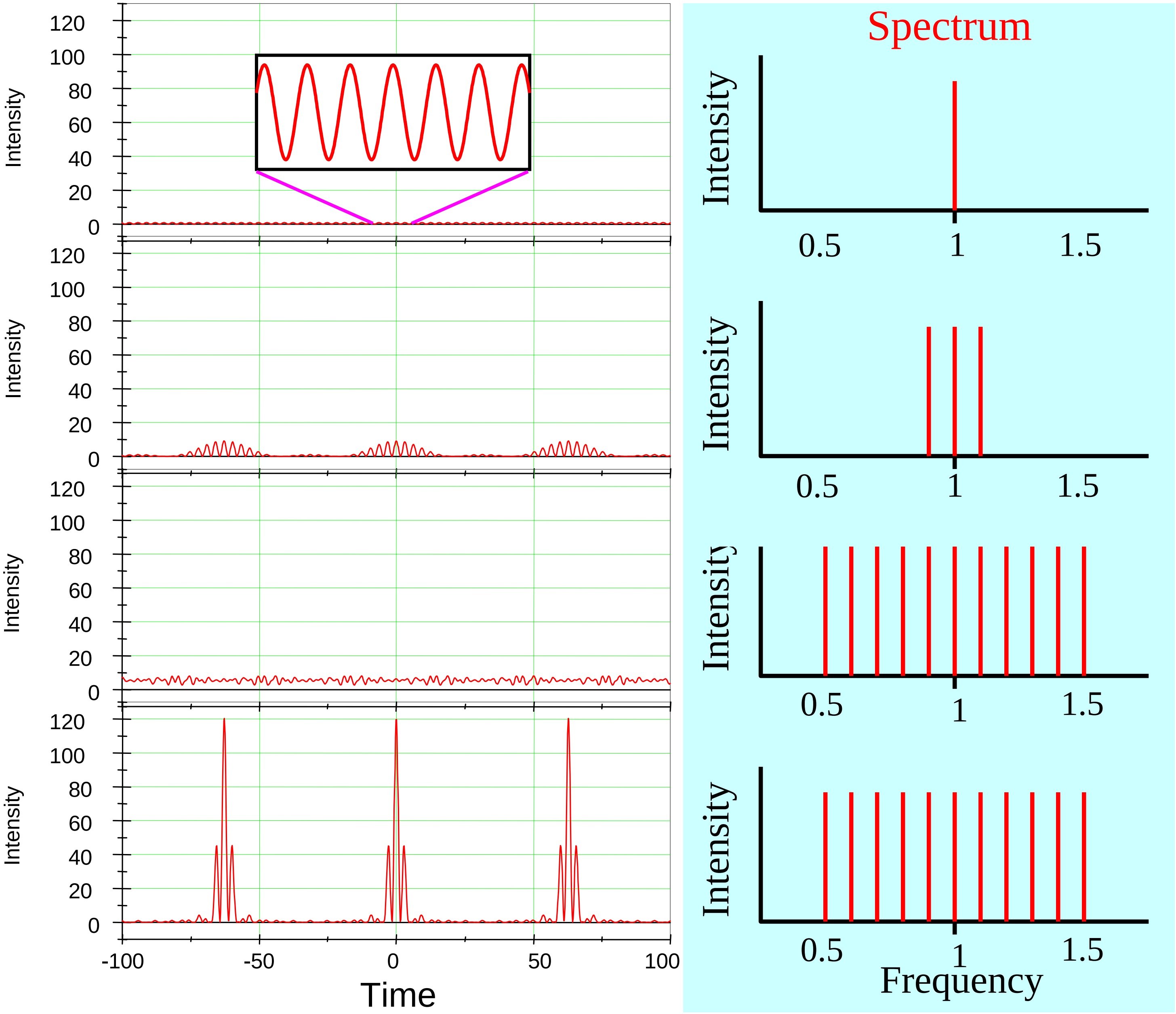
$$\Delta\omega \approx \frac{1}{\tau_p}$$

Single
mode

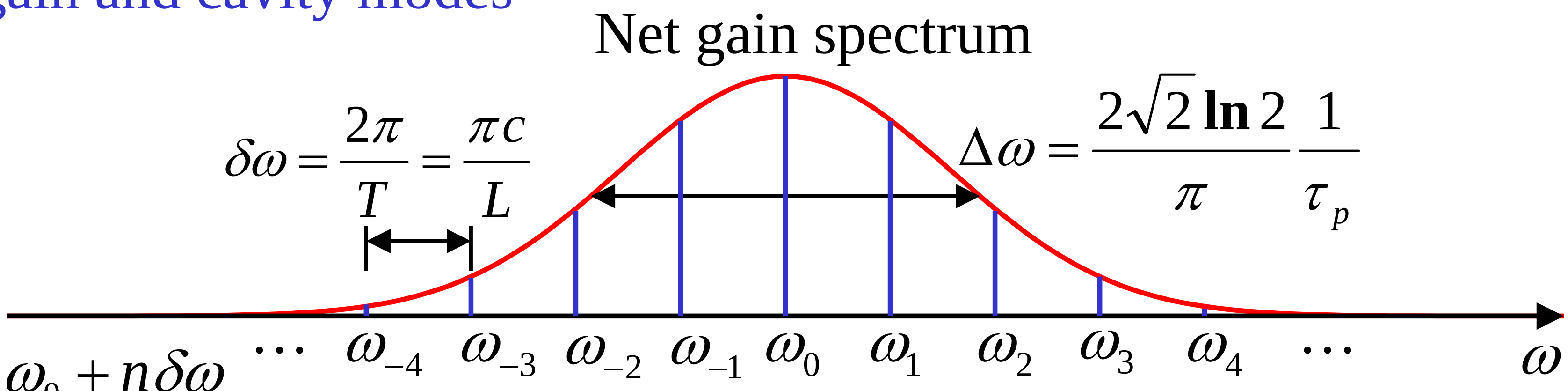
Three
modes
in phase

Eleven modes
random phases

Eleven modes
same phase



Net gain and cavity modes



$$\delta\omega = \frac{2\pi}{T} = \frac{\pi c}{L}$$

$$\Delta\omega = \frac{2\sqrt{2}\,\mathbf{ln}\,2}{\pi}\frac{1}{\tau_p}$$

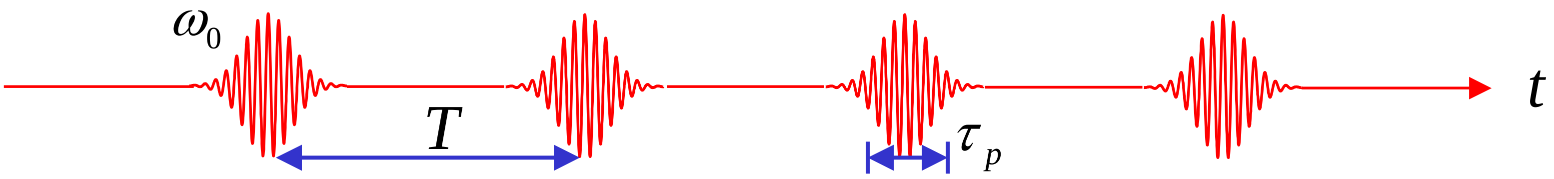
$$\omega_n = \omega_0 + n\delta\omega$$

Total electric field with phase-locked modes: $E(t) = \sum_n E_n e^{i\omega_n t}$

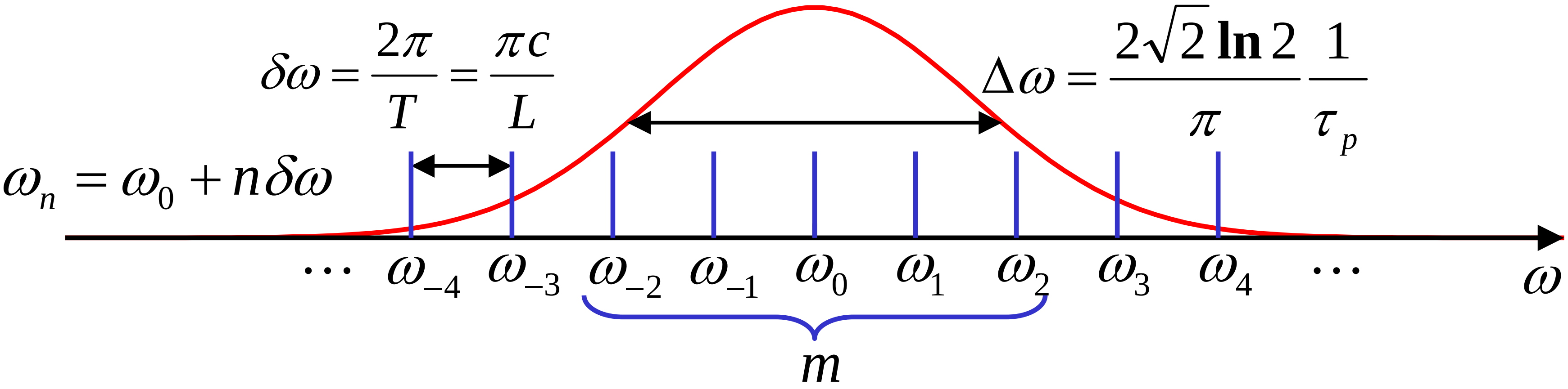
If the gain spectrum is a Gaussian distribution,

$$E_n = E(\omega_n) = E_0\,\mathbf{exp}\left[-4\,\mathbf{ln}\,2\left(\frac{\omega_n - \omega_0}{\Delta\omega}\right)^2\right] = E_0\,\mathbf{exp}\left[-4\,\mathbf{ln}\,2\left(\frac{n\delta\omega}{\Delta\omega}\right)^2\right]$$

$$\Rightarrow \quad E(t) = e^{i\omega_0 t}\sum_n E(\omega_n)e^{in\delta\omega t} = E_0 e^{i\omega_0 t}\sum_m \mathbf{exp}\left[-2\,\mathbf{ln}\,2\left(\frac{t - mT}{\tau_p}\right)^2\right]$$



Net gain spectrum



$$\delta\omega = \frac{2\pi}{T} = \frac{\pi c}{L}$$

$$\Delta\omega = \frac{2\sqrt{2}\,\mathbf{ln}\,2}{\pi}\,\frac{1}{\tau_p}$$

$$\omega_n = \omega_0 + n\delta\omega$$

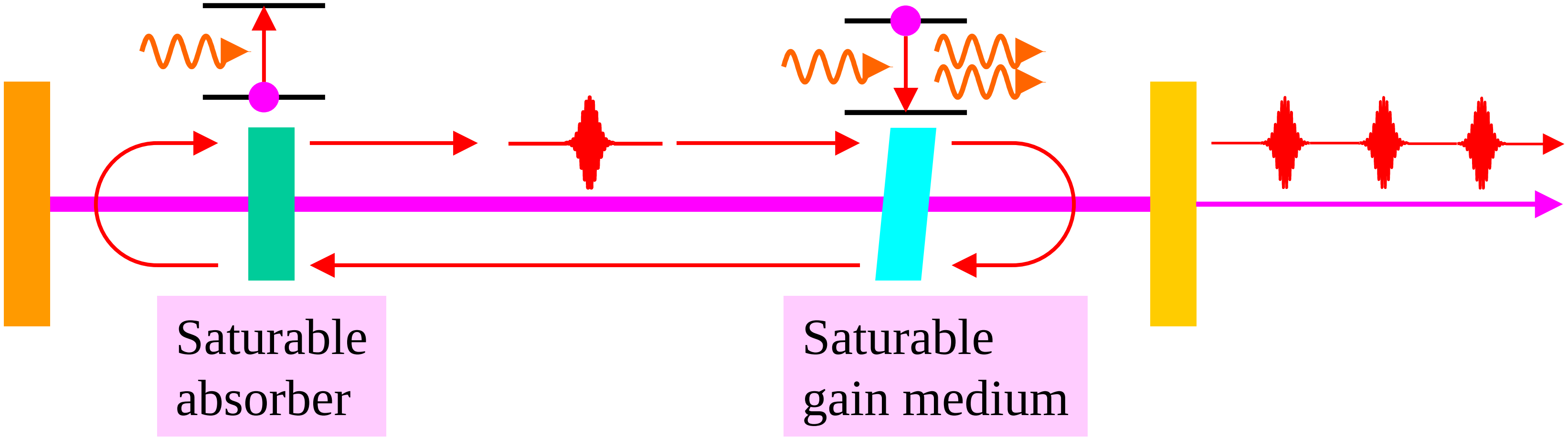
For a laser cavity, *L* = 1.5m:

$$\delta\omega = \frac{2\pi}{T} = \frac{\pi c}{L} = 6.28\times10^{8}\,Hz = 0.628\,GHz, \quad \Delta\lambda \approx \frac{\lambda^2}{2\pi c}\Delta\omega$$

Gain Medium (wavelength)	Bandwidth $\Delta\lambda$ (nm)	Number of modes $m = \Delta\omega / \delta\omega$	Minimum Pulse duration, τ_p
Ar^+-ion (520 nm)	~ 0.007	~ 80	~ 150000 fs
Rubi (694.3 nm)	~ 0.2	~ 1300	~ 6000 fs
Nd:YAG (1064 nm)	~ 10	~ 25000	~ 120 fs
Dye (620 nm)	~ 100	~ 8×10^5	~ 12 fs
Ti:Sapphire (800 nm)	~ 400	~2×10^6	~ 3 fs

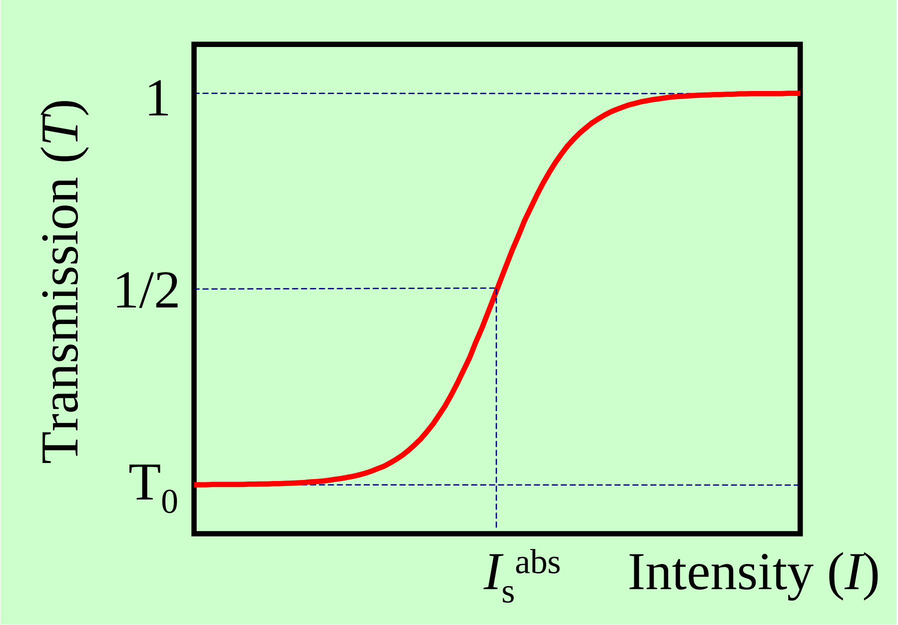
Passive and Hybrid Mode-Locking

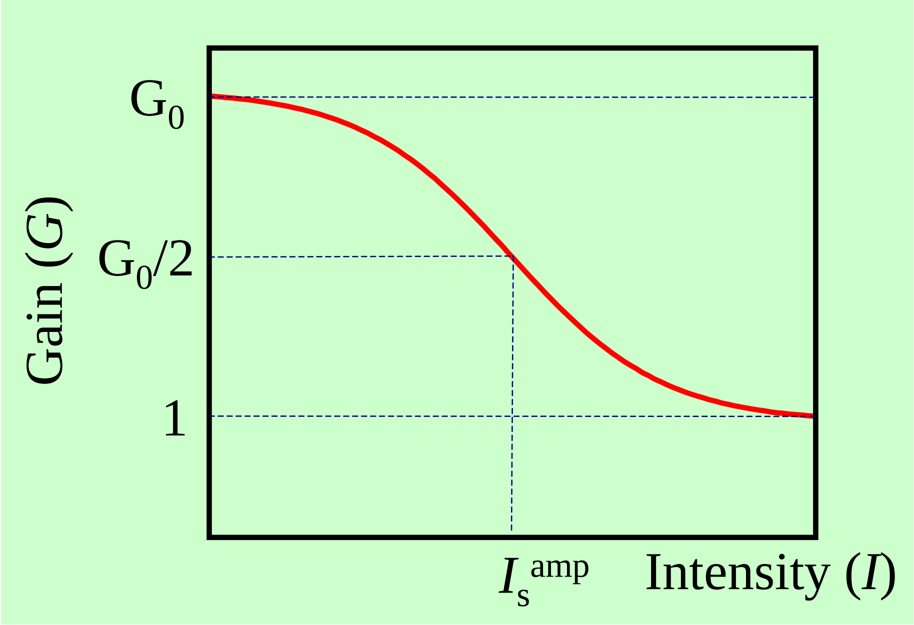
Passive mode-locking is accomplished by interplay between saturable absorber and gain medium without any external modulation.



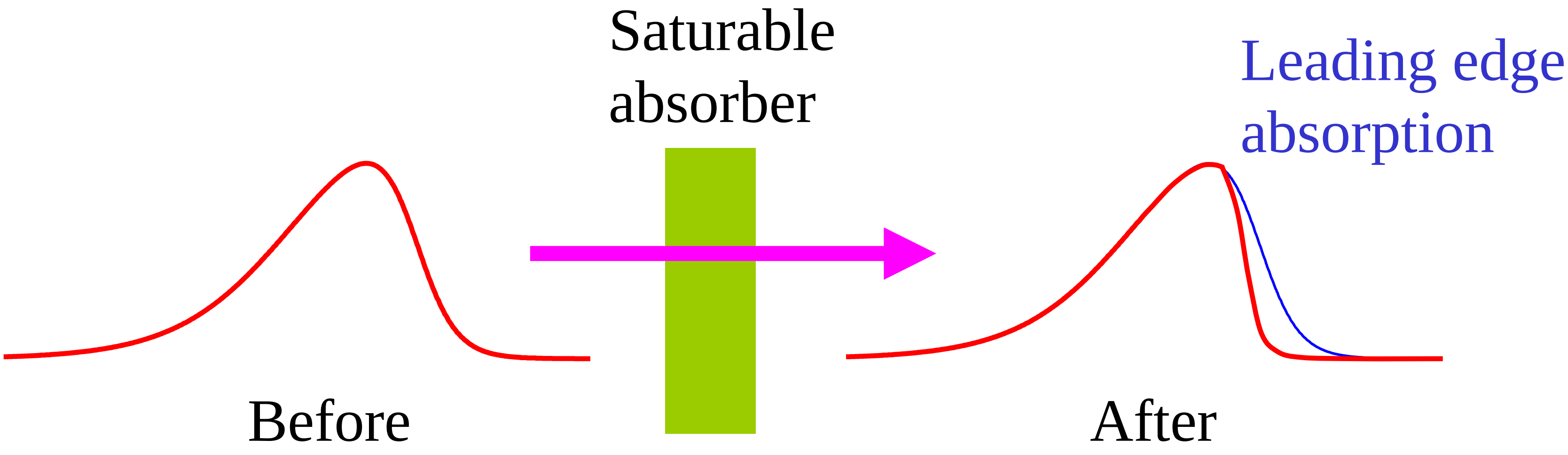
Saturable absorber

Saturable gain medium





Pulse shape modification by saturable absorber and gain medium

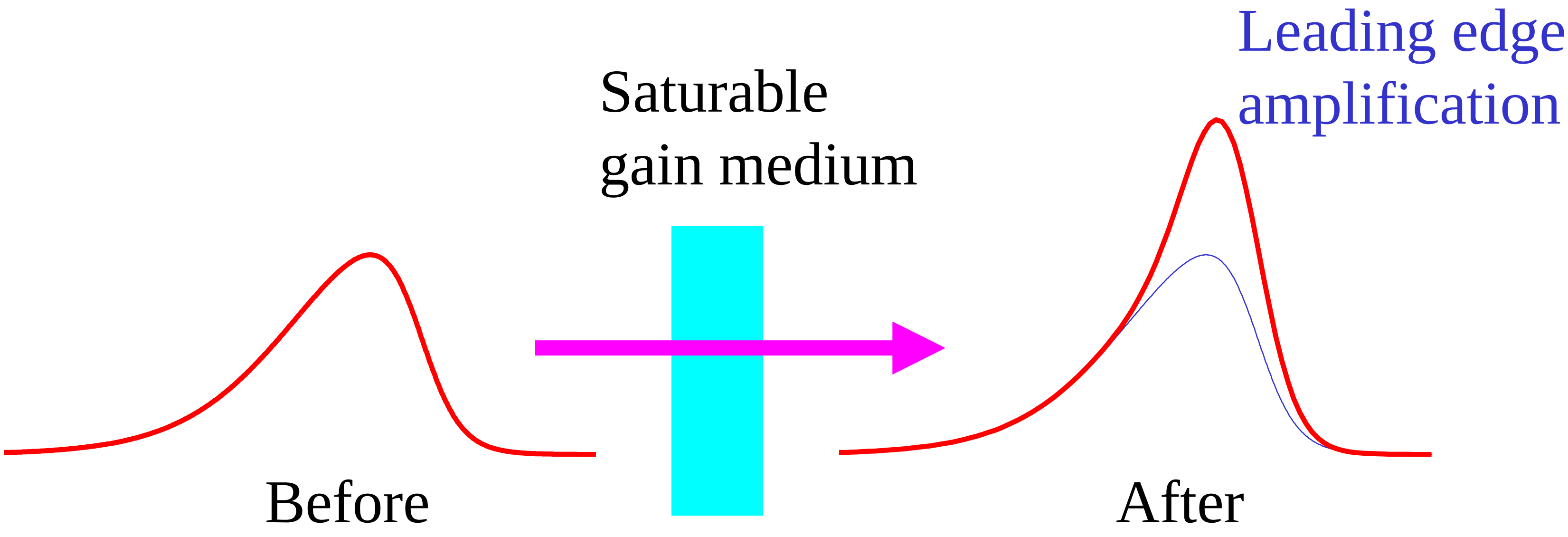


Saturable absorber

Leading edge absorption

Before

After



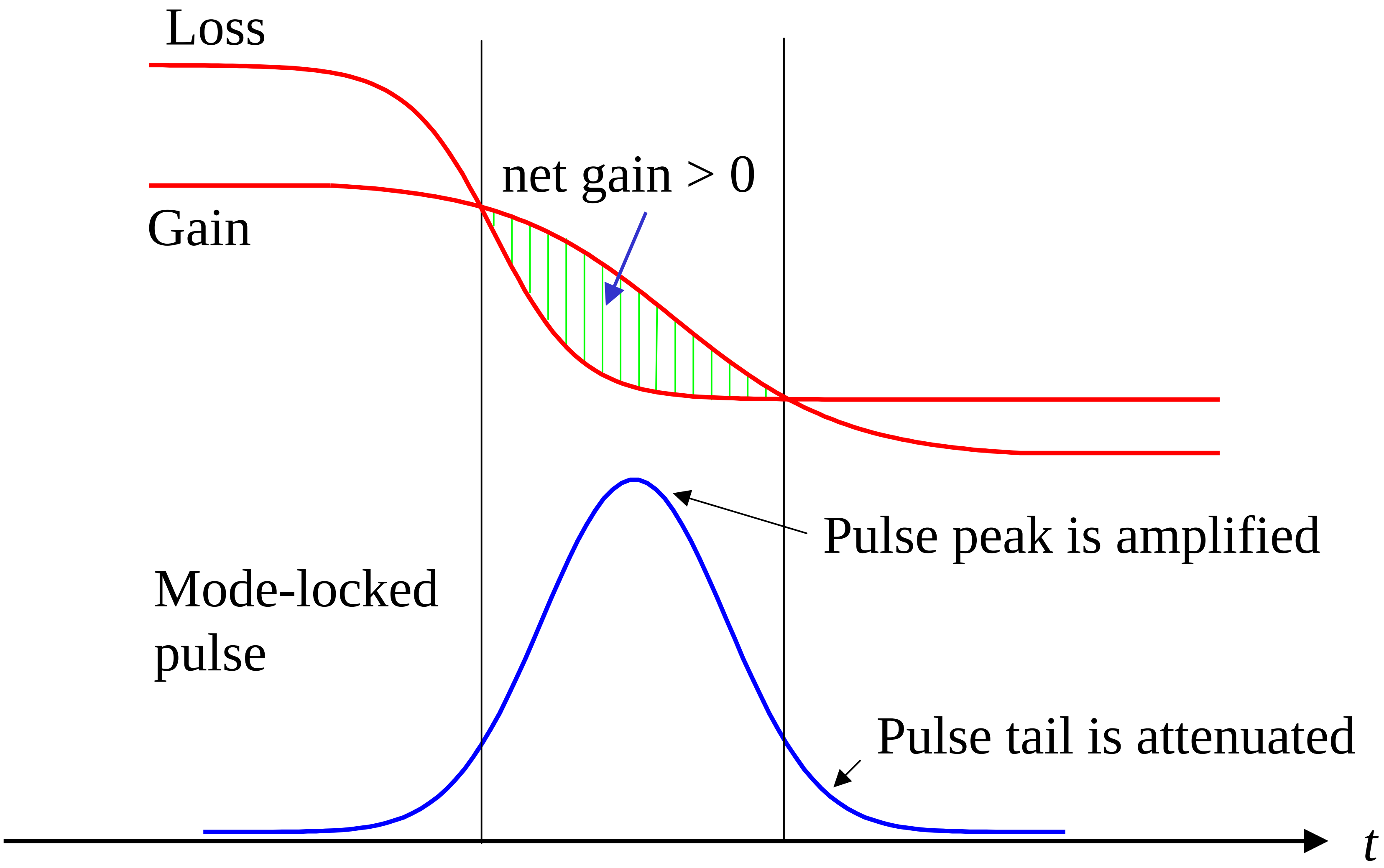
Saturable gain medium

Leading edge amplification

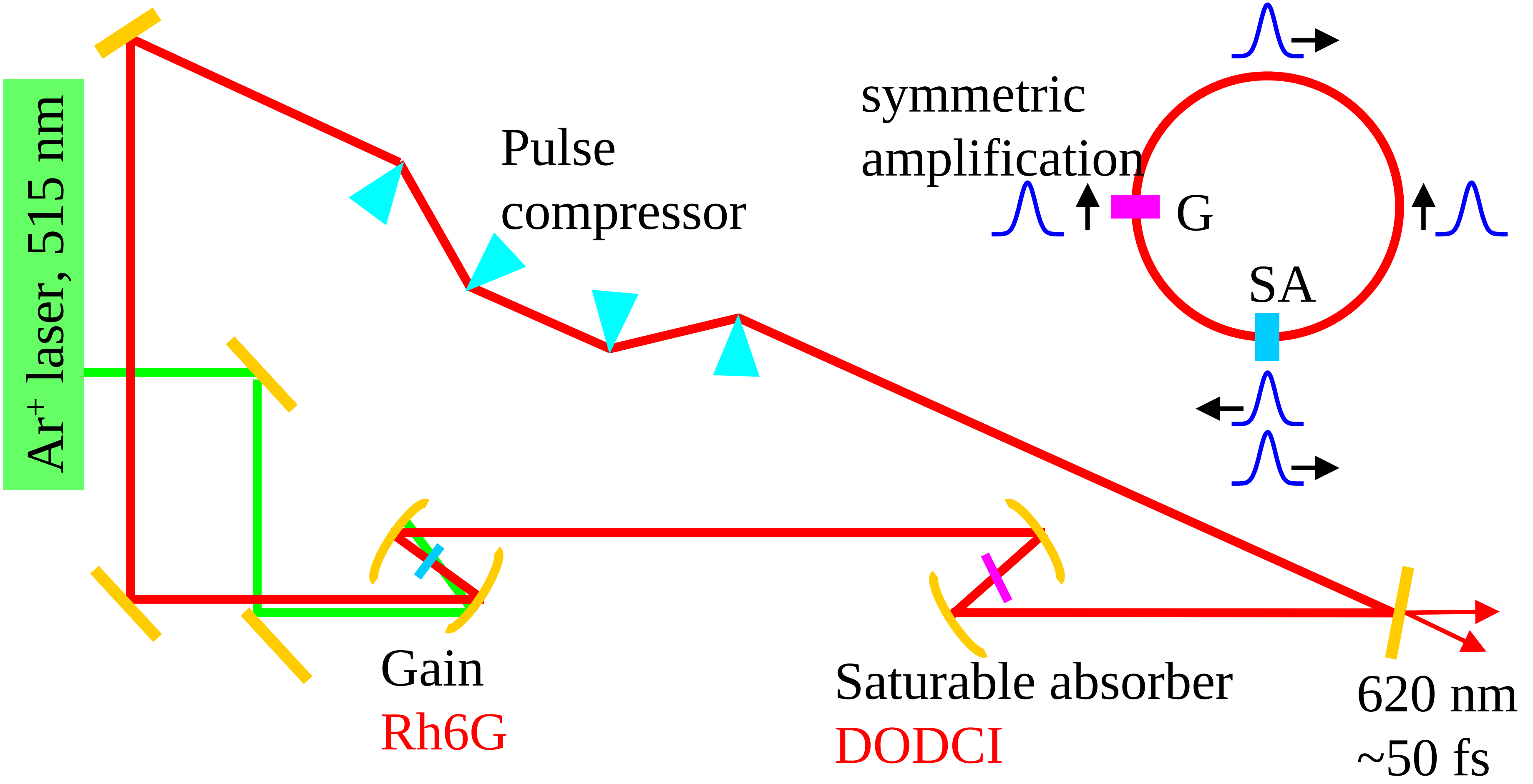
Before

After

Pulse shortening process due to saturation of the absorption and the gain



Colliding-pulse mode-locked dye laser



Ar$^+$ laser, 515 nm

Pulse compressor

symmetric amplification

Gain

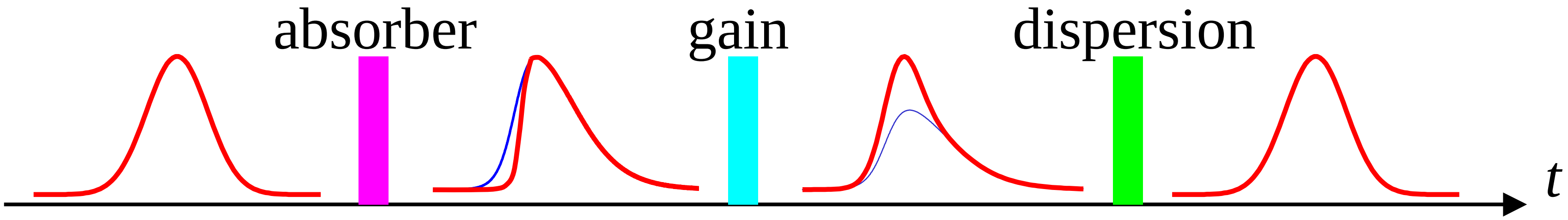
Rh6G

Saturable absorber

DODCI

620 nm

~50 fs



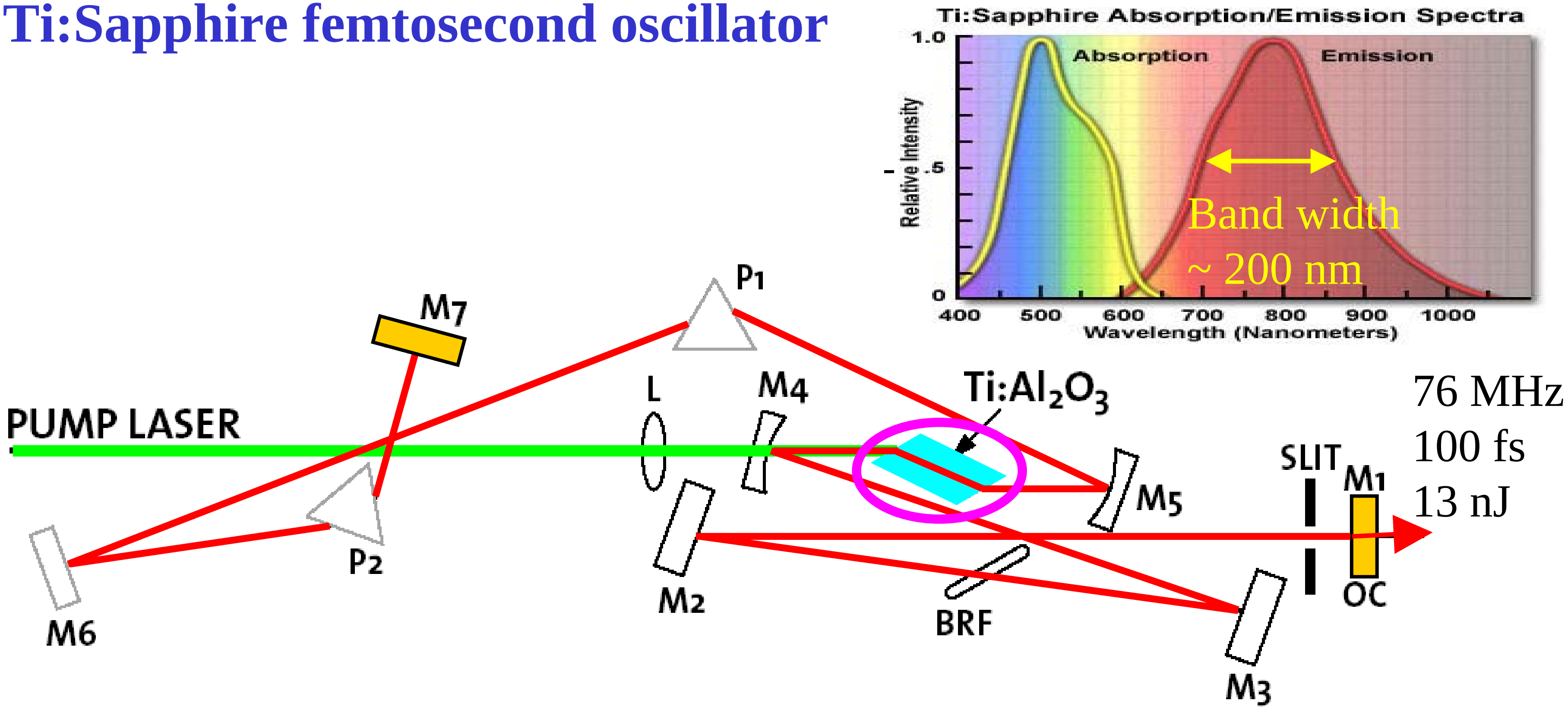
absorber

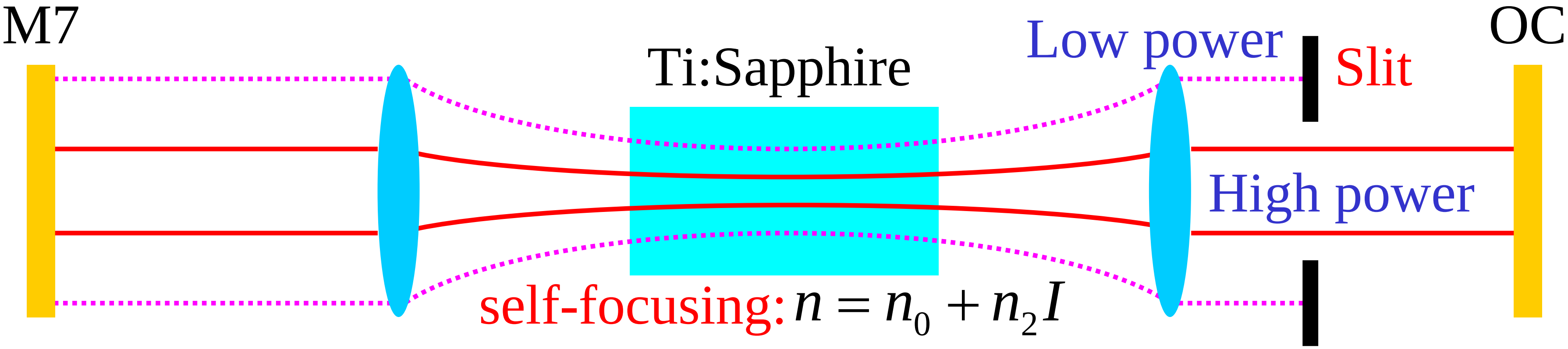
gain

dispersion

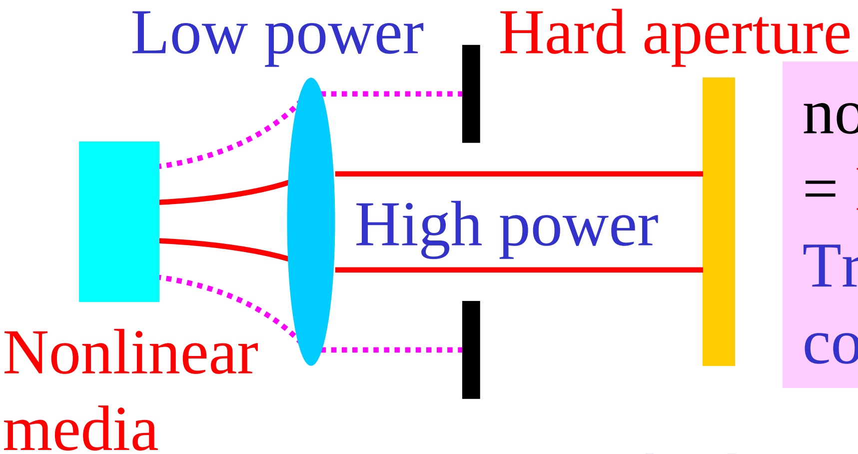
Kerr-Lens Mode-Locking (KLM)

Ti:Sapphire femtosecond oscillator



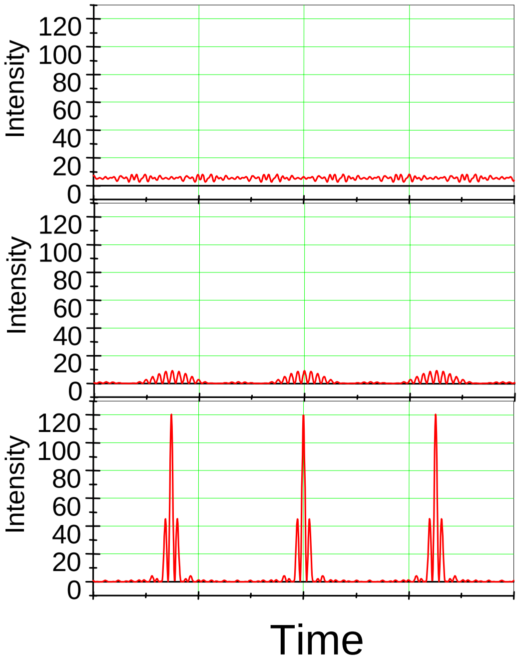


self-focusing: $n = n_0 + n_2 I$



nonlinear medium + hard aperture = Instantaneous saturable absorber : Transmission is lower at low intensity compared to high intensity.

Note: Sometimes a hard aperture is not necessary, because larger overlap between the pump beam and the cavity mode in the gain medium at high intensity leads to larger modal gain.



1. Mechanical shocks (e.g., vibrating starter, jolting mirror) initiate mode-locking.
2. The pulse regime is favored over the continuous regime.
3. Spectral broadening by SPM supports the pulse shortening.