

Question 1.

a) The two measured energies are the $n = 1, 2$ states, so the measurement tells us that:

$$\begin{aligned}\mathcal{P}(E_n) &= |\langle n | \psi(t) \rangle|^2 = |c_n|^2 \\ \mathcal{P}(E_1) &= 0.36 = \frac{9}{25} = \left(\frac{3}{5}\right)^2 \Rightarrow c_1 = \frac{3}{5} e^{i\theta_1} \\ \mathcal{P}(E_2) &= 0.64 = \frac{16}{25} = \left(\frac{4}{5}\right)^2 \Rightarrow c_2 = \frac{4}{5} e^{i\theta_2}\end{aligned}$$

Thus we can construct the original state and the time evolved state:

$$\begin{aligned}|\psi(0)\rangle &= \frac{1}{5} [3e^{i\theta_1} |1\rangle + 4e^{i\theta_2} |2\rangle] \\ |\psi(t)\rangle &= \frac{1}{5} [3e^{i\theta_1} e^{-iE_1 t/\hbar} |1\rangle + 4e^{i\theta_2} e^{-iE_2 t/\hbar} |2\rangle] \\ |\psi(t)\rangle &= \frac{1}{5} [3e^{i\theta_1} e^{-i\frac{3\omega t}{2}} |1\rangle + 4e^{i\theta_2} e^{-i\frac{5\omega t}{2}} |2\rangle] \\ |\psi(t)\rangle &= \frac{1}{5} e^{i\theta_1} e^{-i\frac{3\omega t}{2}} [3|1\rangle + 4e^{i(\theta_2 - \theta_1)} e^{-i\omega t} |2\rangle]\end{aligned}$$

Now use this to find the expectation value of the position.

$$\begin{aligned}\langle x \rangle &= \langle \psi(t) | x | \psi(t) \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \psi(t) | a^\dagger + a | \psi(t) \rangle \\ \langle x \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \frac{1}{5} e^{-i\theta_1} e^{+i\frac{3\omega t}{2}} [3\langle 1 | + 4e^{-i(\theta_2 - \theta_1)} e^{+i\omega t} \langle 2 |] (a^\dagger + a) \frac{1}{5} e^{i\theta_1} e^{-i\frac{3\omega t}{2}} [3|1\rangle + 4e^{i(\theta_2 - \theta_1)} e^{-i\omega t} |2\rangle] \\ \langle x \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \frac{1}{25} [12e^{i(\theta_2 - \theta_1)} e^{-i\omega t} \langle 1 | a | 2 \rangle + 12e^{-i(\theta_2 - \theta_1)} e^{+i\omega t} \langle 2 | a^\dagger | 1 \rangle] \\ \langle x \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \frac{12}{25} [e^{i(\theta_2 - \theta_1)} e^{-i\omega t} \sqrt{2} + e^{-i(\theta_2 - \theta_1)} e^{+i\omega t} \sqrt{2}] \\ \langle x \rangle &= \sqrt{\frac{\hbar}{m\omega}} \frac{24}{25} \cos[\omega t - (\theta_2 - \theta_1)]\end{aligned}$$

This will be minimal when the argument is π , so if this is at $t = 0$, we get

$$\begin{aligned}-(\theta_2 - \theta_1) &= \pi \\ e^{i(\theta_2 - \theta_1)} &= e^{-i\pi} = -1\end{aligned}$$

We can ignore the overall phase of the wave function since it is not measurable and write the state as:

$$|\psi(t)\rangle = \frac{1}{5} [3|1\rangle - 4e^{-i\omega t} |2\rangle]$$

b) Now use this to find the expectation value of the momentum.

$$\begin{aligned}\langle p \rangle &= \langle \psi(t) | p | \psi(t) \rangle = i\sqrt{\frac{m\omega\hbar}{2}} \langle \psi(t) | a^\dagger - a | \psi(t) \rangle \\ \langle p \rangle &= i\sqrt{\frac{m\omega\hbar}{2}} \frac{1}{5} [3\langle 1 | - 4e^{+i\omega t} \langle 2 |] (a^\dagger - a) \frac{1}{5} [3|1\rangle - 4e^{-i\omega t} |2\rangle] \\ \langle p \rangle &= i\sqrt{\frac{m\omega\hbar}{2}} \frac{1}{25} [12e^{-i\omega t} \langle 1 | a | 2 \rangle - 12e^{+i\omega t} \langle 2 | a^\dagger | 1 \rangle] \\ \langle p \rangle &= \sqrt{m\omega\hbar} \frac{24}{25} \sin(\omega t)\end{aligned}$$

c) Now find the expectation value of the energy, noting that the Hamiltonian is diagonal.

$$\begin{aligned}\langle E \rangle &= \langle \psi | H | \psi \rangle \\ |\psi\rangle &= \frac{1}{5} [3|1\rangle - 4e^{-i\omega t} |2\rangle] \\ \langle E \rangle &= \frac{1}{5} [3\langle 1 | - 4e^{-i\omega t} \langle 2 |] H \frac{1}{5} [3|1\rangle - 4e^{-i\omega t} |2\rangle] \\ \langle E \rangle &= \frac{1}{25} [9\langle 1 | H | 1 \rangle + 16\langle 2 | H | 2 \rangle] \\ \langle E \rangle &= \frac{1}{25} [9\frac{3}{2}\hbar\omega + 16\frac{5}{2}\hbar\omega] = \frac{107}{50}\hbar\omega\end{aligned}$$

$$\boxed{\langle E \rangle = \frac{107}{50}\hbar\omega = 2.14\hbar\omega}$$

The same result is obtained with the probability weighting method:

$$\langle E \rangle = \sum_n E_n \mathcal{P}(E_n) = \frac{3}{2}\hbar\omega \frac{9}{25} + \frac{5}{2}\hbar\omega \frac{16}{25} = \frac{107}{50}\hbar\omega$$

Question 2.

$$H \doteq V_0 \begin{pmatrix} 3 & \epsilon & 0 & 0 \\ \epsilon & 3 & 2\epsilon & 0 \\ 0 & 2\epsilon & 5 & \epsilon \\ 0 & 0 & \epsilon & 7 \end{pmatrix}$$

a) For the unperturbed case we have

$$H_0 \doteq V_0 \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

with eigenvalues $E_1 = 3V_0, E_2 = 3V_0, E_3 = 5V_0, E_4 = 7V_0$ and eigenvectors

$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |4\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that $|1\rangle$ and $|2\rangle$ are degenerate and $|3\rangle$ and $|4\rangle$ are nondegenerate.

- b) Let's first do nondegenerate perturbation theory for the $|3\rangle$ and $|4\rangle$ states. First we need to write the perturbation Hamiltonian:

$$H' \doteq V_0 \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 0 & \varepsilon \\ 0 & 0 & \varepsilon & 0 \end{pmatrix}$$

The first-order corrections are

$$E_n^{(1)} = \langle \psi_n^0 | \hat{H}' | \psi_n^0 \rangle$$

$$E_3^{(1)} = \langle \psi_3^0 | \hat{H}' | \psi_3^0 \rangle = 0$$

$$E_4^{(1)} = \langle \psi_4^0 | \hat{H}' | \psi_4^0 \rangle = 0$$

So we need to go to second order for these states

$$E_n^{(2)} = \sum_{k \neq n} \frac{|\langle n | \hat{H}' | k \rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$E_3^{(2)} = \sum_{k \neq 3} \frac{|\langle 3 | \hat{H}' | k \rangle|^2}{E_3^{(0)} - E_k^{(0)}} = \frac{|\langle 3 | \hat{H}' | 1 \rangle|^2}{E_3^{(0)} - E_1^{(0)}} + \frac{|\langle 3 | \hat{H}' | 2 \rangle|^2}{E_3^{(0)} - E_2^{(0)}} + \frac{|\langle 3 | \hat{H}' | 4 \rangle|^2}{E_3^{(0)} - E_4^{(0)}}$$

$$E_3^{(2)} = \frac{|0|^2}{5V_0 - 3V_0} + \frac{|2\varepsilon V_0|^2}{5V_0 - 3V_0} + \frac{|\varepsilon V_0|^2}{5V_0 - 7V_0}$$

$$E_3^{(2)} = \varepsilon^2 V_0 \left(\frac{4}{2} + \frac{1}{-2} \right)$$

$$E_3^{(2)} = \frac{3}{2} \varepsilon^2 V_0$$

$$E_4^{(2)} = \sum_{k \neq n} \frac{|\langle 4 | \hat{H}' | k \rangle|^2}{E_4^{(0)} - E_k^{(0)}} = \frac{|\langle 4 | \hat{H}' | 1 \rangle|^2}{E_4^{(0)} - E_1^{(0)}} + \frac{|\langle 4 | \hat{H}' | 2 \rangle|^2}{E_4^{(0)} - E_2^{(0)}} + \frac{|\langle 4 | \hat{H}' | 3 \rangle|^2}{E_4^{(0)} - E_3^{(0)}}$$

$$E_4^{(2)} = \frac{|0|^2}{7V_0 - 3V_0} + \frac{|0|^2}{7V_0 - 3V_0} + \frac{|\epsilon V_0|^2}{7V_0 - 5V_0}$$

$$E_4^{(2)} = \frac{1}{2} \epsilon^2 V_0$$

Thus the energies to second order are

$$\boxed{E_3 = V_0 \left[5 + \frac{3}{2} \epsilon^2 \right]}$$

$$\boxed{E_4 = V_0 \left[7 + \frac{1}{2} \epsilon^2 \right]}$$

Now look at the perturbation of the degenerate $|1\rangle$ and $|2\rangle$ states. Here we need to diagonalize the perturbation Hamiltonian within that 2x2 space:

$$H' \doteq V_0 \begin{pmatrix} 0 & \epsilon & 0 & 0 \\ \epsilon & 0 & 2\epsilon & 0 \\ 0 & 2\epsilon & 0 & \epsilon \\ 0 & 0 & \epsilon & 0 \end{pmatrix} \Rightarrow H'_{1,2} \doteq V_0 \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}$$

Now diagonalize this

$$\begin{vmatrix} -\lambda & \epsilon V_0 \\ \epsilon V_0 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\epsilon V_0)^2 = 0$$

$$\lambda = \pm \epsilon V_0$$

Now the corrected energies are

$$E_1 = E_1^{(0)} + E_1^{(1)} = 3V_0 + \epsilon V_0$$

$$E_2 = E_2^{(0)} + E_2^{(1)} = 3V_0 - \epsilon V_0$$