Summary of Bessel Functions

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$$x^2 \frac{d^2}{dx^2} J_m(x) + x \frac{d}{dx} J_m(x) + (x^2 - m^2) J_m(x) = 0$$

This is Bessel's equation. Usually $x = k\rho$, where ρ is the radial variable in cylindrical coordinates. J_m is called the "Bessel function of the first kind." The Neumann function N_m, or "Bessel function of the second kind," is also a solution to the same equation. Both also satisfy the general orthogonality and orthonormality relations,

$$(k^2 - l^2) \int_a^b x J_m(kx) J_m(lx) dx = \left[(l \dot{J}_m(lx) J_m(kx) - k \dot{J}_m(kx) J_m(lx)) x \right]_a^b$$

$$\int_a^b x J_m^2(kx) x dx = \left[\frac{1}{2} (x^2 - \frac{m^2}{k^2}) J_m^2(kx) + \frac{x^2}{2} \dot{J}_m^2(kx) \right]_a^b$$

These equations are too general for most applications. In this class we will usually use the Besel functions to expand a potential in some cylindrical region $0 \le \rho \le a$. In this case we simplify with the following replacements:

- Abandon N_m. These functions diverge at the origin.
- Repace $a \to 0$ and $b \to a$.
- Define $k_{mn} = x_{mn}/a$, where x_{mn} is the nth zero of J_m, *i.e.*

  $$J_m(x_{mn}) = 0.$$

They then take the form

$$\int_0^a J_m(k_{mn}\rho) J_m(k_{mn'}\rho) \rho d\rho = \frac{a^2}{2} J_{m\pm 1}^2(x_{mn}) \delta_{nn'}$$

Any function that is not too badly behaved in this region can be expanded in the so-called Fourier-Bessel series.

$$R(\rho) = \sum_{n=1}^{\infty} A_{mn} J_m(k_{mn}\rho)$$

In many respects this is like the Fourier sine series,

$$R(\rho) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}\rho\right)$$

which you can think of as a sum over the "zeros of the sine function," $n\pi/a$. The major difference is that any function $R(\rho)$ can be expanded using Bessel functions in an infinite number of different ways, one for each value of m. By inverting this series we get a completness relation.

$$\frac{\delta(\rho-\rho')}{\rho} = \sum_{n=1}^{\infty} \frac{2J_m(k_{mn}\rho)J_m(k_{mn}\rho')}{a^2 J_{m\pm1}^2(x_{mn})}$$

The Fourier sine series can be generalized so that it applies to an infinite interval. In this case the sums are replaced by integrals. The analogous formulas for Bessel functions are:

$$\frac{\delta(x-x')}{x} = \int_0^{\infty} k\,dk\; J_m(kx)J_m(kx')$$

$$\frac{\delta(k-k')}{k} = \int_0^{\infty} x\,dx\; J_m(kx)J_m(k'x)$$

In addition to J_m and N_m there are other, complex solutions to Bessel's equation:

$$H_m^{(1)} = J_m(x) + iN_m(x)$$

$$H_m^{(2)} = J_m(x) - iN_m(x)$$

These are called Hankel functions, or "Bessel functions of the third kind." Just as J_m and N_m are analogous to sines and cosines, $H_m^{(1)}$ and $H_m^{(2)}$ are analogous to exponentials of the form $e^{\pm i(x-\delta)}/\sqrt{x}$. We will have no use for these, since our potentials are always real functions.

Sometimes one needs solutions to the "modified Bessel equation,"

$$x^2\frac{d^2}{dx^2}R_m(x) + x\frac{d}{dx}R_m(x) - (x^2+m^2)R_m(x) = 0$$

Function	Trig Function	Behavior
J_m, N_m	$\sin(x)$, $\cos(x)$	Real Oscillatory
$H_m^{(1)}$, $H_m^{(2)}$	$e^{\pm ix}$	Complex, oscillatory
I_m	e^x	Regular at 0, singular at ∞.
K_m	e^{-x}	Singular at 0, regular at ∞.

Table 1: Bessel functions and their behavior together with the corresponding trigonometric functions.

This is equivalent to Bessel's equation with x replaced by ix. Accordingly we define

$$I_m(x) = \frac{J_m(ix)}{i^m}$$

$$K_m(x) = \frac{\pi(i)^{m+1}}{2}[J_m(ix) + iN_m(ix)]$$

For large x then

$$I_m \to \frac{1}{\sqrt{x}} e^x$$

$$K_m \to \frac{1}{\sqrt{x}} e^{-x}$$

Since these functions do not oscillate they have no simple orthogonality relation. These properties are summarized in Table 1.

When we separate variables in cylindrical coordinates, we are always faced with a choice for the sign of the separation constant k^2.

$$\ddot{Z} = \pm k^2 Z$$

$$\rho^2 \ddot{R} + \rho \dot{R} + (\pm k^2 \rho^2 - m^2) R = 0$$

Either one takes in upper sign, in which case Z is exponential and R oscillates, or one takes the lower sign in which case Z oscillates and R is exponential.

The following recursion relations are often useful:

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$$

$$J_{m-1}(x) - J_{m+1}(x) = 2\frac{d}{dx} J_m(x)$$

The functions J_m, N_m, $H_m^{(1)}$, and $H_m^{(2)}$ all satisfy the formulas above. The Modified Bessel function I_m and K_m satisfy the following:

$$I_{m-1}(x) - I_{m+1}(x) = -\frac{2m}{x} I_m(x)$$

$$I_{m-1}(x) + I_{m+1}(x) = 2\frac{d}{dx} I_m(x)$$

$$K_{m-1}(x) - K_{m+1}(x) = -\frac{2m}{x} K_m(x)$$

$$K_{m-1}(x) + K_{m+1}(x) = -2\frac{d}{dx} K_m(x)$$