Simpson's rule derivation:

Note Title 2/20/2006

① $$f(x) = \alpha x^2 + \beta x + \gamma$$

$$\int f(x)\,dx = \frac{\alpha}{3}\left(x_{i+1}^3 - x_{i-1}^3\right) + \frac{\beta}{2}\left(x_{i+1}^2 - x_{i-1}^2\right) + \gamma(x_{i+1} - x_{i-1})$$

$$\begin{cases} f(x_{i-1}) = \alpha x_{i-1}^2 + \beta x_{i-1} + \gamma \\ f(x_i) = \alpha x_i^2 + \beta x_i + \gamma \\ f(x_{i+1}) = \alpha x_{i+1}^2 + \beta x_{i+1} + \gamma \end{cases}$$

$$f(x_{i+1}) - f(x_i) = \alpha\left(x_{i+1}^2 - x_i^2\right) + \beta(x_{i+1} - x_i) =$$
$$= \alpha h (x_{i+1} + x_i) + \beta h$$

$$f(_{i-1}) - f(x_i) = \alpha\left(x_{i-1}^2 - x_i^2\right) + \beta(x_{i-1} - x_i) =$$
$$= -\alpha h (x_{i-1} + x_i) - \beta h$$

$$f(x_{i+1}) + f(x_{i-1}) - 2f(x_i) = \alpha h[x_{i+1} - x_{i-1}] = 2\alpha h^2$$

$$\alpha = \frac{f(x_{i+1}) + f(x_{i-1}) - 2f(x_i)}{2h^2}$$

$$x_i = h, \quad x_{i+1} = 2h$$

$$\beta h_1 = -\frac{f(x_{i+1}) + f(x_{i-1}) - 2f(x_i)}{2\cancel{h}} \cancel{h} - f(x_{i-1}) - f(x_i) =$$

$$= -\frac{f(x_{i+1})}{2} + 2f(x_i) - \frac{3}{2} f(x_{i-1})$$

$$\gamma = f(x_{i-1})$$

$$\int = \frac{\alpha}{3} 8h^3 + \frac{\beta}{2} \cdot 4h^2 + 2\gamma h = h \left[\frac{f(x_{i+1}) + f(x_{i-1}) - 2f(x_i)}{} \frac{4}{3} + \right.$$

$$\left. + 2\left(-\frac{f(x_{i+1})}{2} + 2f(x_i) - \frac{3}{2} f(x_{i-1})\right)\right] + 2f(x_{i-1}) \Big] =$$

$$= h\left[f(x_{i+1}) \left\{ \frac{4}{3} - 1 \right\} + f(x_i) \left\{ -\frac{8}{3} + 4 \right\} + f(x_{i-1}) \left\{ \frac{4}{3} + 2 - 3 \right\} \right.$$

$$= h\left[-f(x_{i+1}) \frac{1}{3} + f(x_i) \frac{4}{3} + f(x_{i-1}) \cdot \frac{1}{3} \right.$$